GLUING TECHNIQUES AND ENVELOPES OF DISC FUNCTIONALS ON ALMOST COMPLEX MANIFOLDS

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Abstract. We establish plurisubharmonicity of the envelope of Poisson and Lelong functionals on almost complex manifolds. That is, we generalize the corresponding results for complex manifolds and almost complex manifolds of complex dimension two. We also provide some applications to the regularization of \( J \)-plurisubharmonic functions and to the characterization of compact \( \text{Psh}_J \)-hulls by pseudoholomorphic discs.

Let \( \mathbb{D} = \{ \zeta \in \mathbb{C}; |\zeta| < 1 \} \) denote the open unit disc in the complex plane. Given a smooth almost complex manifold \((M, J)\) we denote by \( \mathcal{O}_J(\overline{\mathbb{D}}, M) \) the set of \( J \)-holomorphic discs in \( M \), i.e., the set of smooth maps \( u: \mathbb{D} \to M \) that are \( J \)-holomorphic in some neighborhood of \( \overline{\mathbb{D}} \). A disc functional on \( M \) is a function

\[
H_M: \mathcal{O}_J(\overline{\mathbb{D}}, M) \to \mathbb{R} = [-\infty, +\infty].
\]

Let \( p \in M \). The envelope of \( H_M \) is the function \( EH_M: M \to \mathbb{R} \) defined by

\[
EH_M(p) = \inf \left\{ H_M(u); u \in \mathcal{O}_J(\overline{\mathbb{D}}, M), u(0) = p \right\}.
\]

Given an upper semi-continuous function \( f \) defined on \( M \), the associated Poisson functional is defined by

\[
P_f(u) = \frac{1}{2\pi} \int_0^{2\pi} f(u(e^{it})) \, dt, \quad u \in \mathcal{O}_J(\overline{\mathbb{D}}, M).
\]

We prove the following theorem.

**Theorem 1.** Let \( f: M \to \mathbb{R} \cup \{-\infty\} \) be an upper semi-continuous function defined on a smooth almost complex manifold \((M, J)\). Then \( EP_f \) is \( J \)-plurisubharmonic on \( M \) or identically \(-\infty\).

The envelope \( EP_f \) of the Poisson functional is the largest plurisubharmonic minorant of the upper semi-continuous function \( f \). In the case when \( M = \mathbb{C}^n \) this was proved by Poletsky [17, 18] and by Bu and Schachermayer [1]. The result was extended to some complex manifolds by Lárusson and Sigurdsson [11, 12], and to all complex manifolds by Rosay [21, 22]. Using the method
of sprays Drinovec-Drnovšek and Forstnerič extended the result to locally irreducible complex spaces [5].

The Lelong functional associated to a nonnegative real function $\alpha$ on $M$ is defined by

$$(2) \quad L_\alpha(u) = \sum_{\zeta \in \mathbb{D}} \alpha(u(\zeta)) \log |\zeta|, \quad u \in \mathcal{O}_J(\mathbb{D}, M).$$

Given a $J$-plurisubharmonic function $f \in \text{Psh}_J(M)$ and a point $p \in M$ we denote by $\nu_f(p) \in [0, +\infty]$ its Lelong number at $p$: in any local coordinate system $\psi$ on $M$, with $\psi(p) = 0$, we have

$$\nu_f(p) = \lim_{r \to 0} \sup_{|\psi(q)| \leq r} f(q) \log r.$$  

(We consider the constant function $f = -\infty$ as $J$-plurisubharmonic and set $\nu_{-\infty} = +\infty$.) Given a nonnegative function $\alpha : M \to \mathbb{R}_+$, we consider the corresponding extremal function on $M$

$$(3) \quad v_\alpha = \sup \{ f \in \text{Psh}_J(M); f \leq 0, \nu_f \geq \alpha \}.$$  

**Theorem 2.** Let $\alpha : M \to \mathbb{R}_+$ be defined on a smooth almost complex manifold $(M, J)$. Then $EL_\alpha$ is $J$-plurisubharmonic and equals to $v_\alpha$.

This disc formula was first obtained by Poletsky [18]. On manifolds it was proved by Lárusson and Sigurdsson [11, 12] and for locally irreducible complex spaces by Drinovec-Drnovšek and Forstnerič [6].

Both theorems were already proved for almost complex manifolds, but only in case of complex dimension two [10, 7]. The missing step was a method of attaching a $J$-holomorphic disc to a real torus. Given an embedded pseudoholomorphic disc $u$, we associate to $u$ a real 2-dimensional torus formed by the boundary circles of discs centered at the boundary points $u(\zeta), \zeta \in \partial \mathbb{D}$. We seek a disc centered at $u(0)$ and approximately attached to such torus. In the low dimensional case such a method is presented in [2, 24]. We give here a new, dimensional free solution to this problem (see Theorem 7) based on gluing techniques described in [14, 15] inspired by [4].

The plan of the paper is as follows. In §1 we introduce and illustrate the gluing techniques and tools that will be used. Section §2 contains our main result on approximately attaching a disc to a torus. In §3 we prove Theorem 1 and Theorem 2. Finally we provide in §4 some applications of Theorem 1 to the regularization of $J$-plurisubharmonic functions and to the characterization of compact $\text{Psh}_J$-hulls by pseudoholomorphic discs.
1. THE BASIC GLUING

An almost complex structure $J$ on a real smooth manifold $M$ is a $(1, 1)$ tensor field which satisfies $J^2 = -Id$. We suppose that $J$ is smooth. The pair $(M, J)$ is called an almost complex manifold. We denote by $J_{st}$ the standard integrable structure on $\mathbb{C}^n$ for every $n \in \mathbb{N}$. A differentiable map $u : (M', J') \rightarrow (M, J)$ between two almost complex manifolds is $(J', J)$-holomorphic if $J'(u(p)) \circ d_p u = d_p u \circ J'(p)$, for every $p \in M'$. When $M'$ equals to an open disc $\Delta \subset \mathbb{C}$ the map $u$ is called a $J$-holomorphic disc.

Let $u : \Delta \rightarrow M$ and $v \in \mathbb{C}$. We define

$$\partial_J u(v) = \frac{1}{2} (du(v) + J(u)du(iv))$$

and

$$\overline{\partial_J} u(v) = \frac{1}{2} (du(v) - J(u)du(iv)).$$

Note that $u$ is a $J$-holomorphic disc if and only if $\partial_J u = 0$. We denote by $u : \overline{\Delta} \rightarrow M$ the discs that are $J$-holomorphic on some neighborhood of $\Delta$.

In local coordinates $z \in \mathbb{C}^n$ an almost complex structure $J$ is represented by a $\mathbb{R}$-linear operator satisfying $J(z)^2 = -I$. Assume that $J + J_{st}$ is invertible along a disc $u$. The above condition can be rewritten in the form

$$(4) \quad u_\zeta + A(u)\overline{\zeta} = 0,$$

where $\zeta = x + iy$ and $A(z)(v) = (J_{st} + J(z))^{-1}(J(z) - J_{st})(\overline{v})$ is a complex linear endomorphism for every $z \in \mathbb{C}^n$. Hence $A$ can be considered as a $n \times n$ complex matrix of the same regularity as $J(z)$ acting on $v \in \mathbb{C}^n$. We call $A$ the complex matrix of $J$.

1.1. THE LINEARIZED EQUATION. One of the key steps in this paper is understanding the linearized equation (4) for $A(u) = 0$, that is, the equation

$$v_\zeta + B_1 v + B_2 \overline{v} = 0,$$

where $B_1$ and $B_2$ are smooth matrix functions on $\Delta$ arising from the derivatives of $A$. A rather complete theory for these systems in the scalar case was given by Vekua [25]. He named its solutions the generalized analytic functions. In the same spirit, generalized analytic vectors were introduced by Pascali [16]. In our case, the crucial ingredient is [23, Theorem 3.1]. Let us briefly explain its proof and applications here.
Let $p > 2$ and $k \in \mathbb{N}$. We denote by $L^p$ and $W^{k,p}$ the classical Lebesgue and Sobolev spaces. We introduce the Cauchy-Green operator

$$T(f)(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\zeta)}{z-\zeta} \, dx \, dy.$$ 

We need two of its properties. Firstly, $T$ solves the usual $\bar{\partial}$-equation and secondly, $T$ is a bounded operator mapping the space $L^p(\mathbb{D}, \mathbb{R}^{2n})$ into the space $W^{1;p}(\mathbb{D}, \mathbb{R}^{2n})$ [25]. Moreover, \begin{equation}
\Phi(v) = v + T(B_1 v + B_2 \bar{v})
\end{equation}
is a Fredholm map giving a correspondence between the set of generalized analytic and the usual holomorphic vector functions of the same Lebesgue class. However, $\Phi$ may have a non-trivial kernel. Hence, in order to obtain bijectivity we correct it by adding a small linear part arising from the adjoint operator of $\Phi$. That is, there exists a linear operator $L$ such that \begin{equation}
\tilde{\Phi}(v) = v + T(B_1 v + B_2 \bar{v}) + L(v)
\end{equation}
is bijective, $L(v) = 0$ and $\tilde{\Phi}^{-1}: L^p(\mathbb{D}, \mathbb{R}^{2n}) \rightarrow L^p(\mathbb{D}, \mathbb{R}^{2n})$ is bounded.

We use a slightly improved version of this theory. First, note that the transform $T$ may be defined on any disc $\Delta$ in $\mathbb{C}$. Moreover, take a finite set of points $a_j \in \Delta$ and define $P$ to be a polynomial satisfying $P(a_j) = T(f)(a_j)$. If we define $\Phi_P$ as in (5) but using $T_P(f)(z) = T(f)(z) - P(z)$ instead of $T$ it satisfies $\Phi(v)(a_j) = v(a_j)$. Next, we perturb $\Phi_P$ with a small, $J_{sl}$-holomorphic perturbation $L_P$ such that $L_P(v)(a_j) = 0$. To find such a deformation one has to find an operator adjoint to $\Phi_P$ and follow the original proof. (See [7, Lemma 2] for $a_1 = 0$ and $a_2 = b \neq 0$ on $\mathbb{D}$.)

This modified version of the theory allows to deform $J$-holomorphic discs but fixing some selected points, see for instance [7, Proposition 5]. Moreover, if we use it in the proof of [10, Proposition 2] we get the following statement.

**Proposition 3.** Let $a_j \in \Delta$ be a finite set of points and $w: \overline{\Delta} \rightarrow \mathbb{R}^{2n}$ a smooth map. Let $J$ be a smooth almost complex structure defined in a neighborhood of the image $w(\overline{\Delta})$ such that $\det(J + J_{sl}) \neq 0$. Let $A$ be the complex matrix of $J$. For every $\varepsilon > 0$ one can find $\delta > 0$ such that if

$$\|w_\zeta + A(w)\bar{w}_\zeta\|_{L^p(\Delta, \mathbb{R}^{2n})} < \delta$$

there exists a $J$-holomorphic disc $\tilde{w}: \overline{\Delta} \rightarrow \mathbb{R}^{2n}$ such that $\tilde{w}(a_j) = w(a_j)$ and

$$\|w - \tilde{w}\|_{W^{1,p}(\Delta, \mathbb{R}^{2n})} < \varepsilon.$$
Note that in the original reference the result is stated in term of $L^\infty$-norms. However, the proof goes through under the present assumptions.

1.2. The operator $D_u$. Following [15], we define an elliptic operator of order 1 that will play a crucial role in our construction of a $J$-holomorphic disc approximately attached to a torus. Let $(M, J)$ be an almost complex manifold endowed with an arbitrary smooth $J$-invariant Riemannian metric $g$. Let $\nabla$ be the Levi-Civita connection of $g$. We define $J$-compatible linear connection by

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J) Y.$$ 

Let $\Delta \subset \mathbb{C}$ be an open disc and let $p > 2$. Let $u \in W^{1,p}(\Delta, M)$ and let $\xi \in W^{1,p}(\Delta, u^*TM)$. The map $\exp_u(\xi)$ is contained in a neighborhood of $u$ in $W^{1,p}(\Delta, M)$. For $z \in \Delta$, we consider the geodesic $\alpha_z(t) = \exp_u(t\xi(z))$. Recall that a vector field $X$ along $\alpha_z$ is horizontal if it satisfies the first order linear equation $\tilde{\nabla}_{\dot{\alpha}_z} X = 0$. The map $X(0) \mapsto X(1)$ therefore defines an isomorphism between $T_{\alpha_z(0)}M$ and $T_{\alpha_z(1)}M$. We consider the corresponding bundle isomorphism

$$\Phi_u(\xi) : u^*TM \to \exp_u(\xi)^*TM.$$ 

We define $F_u : W^{1,p}(\Delta, u^*TM) \to L^p(\Delta, \Lambda^{0,1} \otimes u^*TM)$ by

$$(7) \quad F_u(\xi) = \Phi_u(\xi)^{-1}(\overline{\partial}_J \exp_u(\xi)).$$ 

Note that $F_u(\xi) = 0$ if and only if $\exp_u(\xi)$ is a $J$-holomorphic disc. We are interested in the linearization $D_u$ of $F_u$ at the origin

$$D_u = d_0F_u : W^{1,p}(\Delta, u^*TM) \to L^p(\Delta, \Lambda^{0,1} \otimes u^*TM)$$ 

We have

$$(8) \quad D_u \xi = \frac{1}{2} (\nabla \xi + J(u)\nabla \xi \circ J_u) - \frac{1}{2} J(u)(\nabla \xi J)(u)\partial_J u = \eta.$$ 

The differential operator $D_u$ is elliptic. It was mostly studied in the case of compact Riemann surfaces or discs attached to a given totally real submanifold of $M$ (see [14, 15]). We use section §1.1 to prove the following proposition.

**Proposition 4.** Let $u \in W^{1,p}(\Delta, M)$ be an embedded $J$-holomorphic disc. Then $D_u$ is Fredholm and has a bounded right inverse $Q_u$.

**Proof.** Since $u$ is an embedding one can find a local chart around its image. Hence we can restrict to the case when $M = \mathbb{R}^{2n}$. Moreover, by a change of
coordinates one can assume that the structure is standard along \( u(\Delta) \) (see e.g. the appendix in [9]). Let \( x + iy \in \Delta \). If we evaluate (8) in \( \frac{\partial}{\partial x} \) we get
\[
\frac{1}{2} \left( \frac{\partial \xi}{\partial x} + J(u) \frac{\partial \xi}{\partial y} + dJ(u)(\xi) \frac{\partial u}{\partial y} \right) = \eta \left( \frac{\partial}{\partial x} \right). 
\]
Similarly to (4) we can rewrite this into an equivalent form
\[
\Phi(\xi) = \xi + A(u)\overline{\xi} + B_1(u)\xi + B_2(u)\overline{\xi} = \tilde{\eta}, 
\]
where the smooth complex matrix functions \( B_1 \) and \( B_2 \) arise from the derivatives of \( J \) along \( u \) and \( \tilde{\eta} \in L^p(\Delta, \mathbb{C}^n) \) arises from \( \eta \). Since \( J(u) = J_{st} \) we have \( A(u) = 0 \). Hence, for \( \Phi \) defined as in (6) the vector function \( \xi = \Phi^{-1}(\tilde{\eta}) \) solves the above equation and suggests a bounded right inverse to the operator \( \Phi \). The rest follows by the fact that \( \overline{\partial}_J(u) \) is anti complex linear. □

**Remark 1.** Let \( u, v \in W^{1,p}(\Delta, M) \) be such that \( D_u \) and \( D_v \) are both Fredholm. Assume also that \( D_u \) surjective and that the discs \( u \) and \( v \) are close enough with respect to the \( W^{1,p}(\Delta, M) \)-topology. Then both \( D_u \) and \( D_v \) admit right inverses \( Q_u \) and \( Q_v \), and close to each other.

**Remark 2.** Let us fix a finite set of points in \( a_j \in \Delta \). The proposition remains true for \( D_u \) mapping from \( X = \{ \xi \in W^{1,p}(\Delta, u^*TM); \xi(a_j) = 0 \} \) to \( Y = \{ \eta \in L^p(\Delta, \Lambda^{0,1} \otimes u^*TM); \eta(u(a_j)) = \overline{\partial}_J u(0) \} \). Indeed, one only needs to use the normalized Cauchy-Green operator \( T_P \) from §1.1.

1.3. Gluing two discs. We include here a simple proposition. It is is not needed in the sequel, but it helps in understanding the basic idea behind the gluing techniques developed in [14, 15].

**Proposition 5.** Let \( \Delta_1, \Delta_2 \subset \mathbb{C} \) be two intersecting discs and assume that \( u_j \in W^{1,p}(\Delta_j, M) \) are two \( J \)-holomorphic embeddings such that for a sufficiently small \( \varepsilon < 0 \) we have
\[
\|u_1 - u_2\|_{W^{1,p}(\Delta_1 \cap \Delta_2, M)} < \varepsilon.
\]
Then there exists a \( J \)-holomorphic map \( h: \Delta_1 \cup \Delta_2 \rightarrow M \) whose image is close to the connected sum of sets \( u_1(\Delta_1) \) and \( u_2(\Delta_2) \).

**Proof.** Since \( u_1 \) is embedded its image fits into a chart \( f_1: \Omega_1 \rightarrow \mathbb{R}^{2n} \). Assume that \( \varepsilon > 0 \) is small enough so that \( u_2(\Delta_1 \cap \Delta_2) \subset \Omega_1 \). Let \( \chi: \Delta_1 \cup \Delta_2 \rightarrow [0, 1] \) be a smooth cut-off function such that \( \chi = 1 \) on \( \Delta_1 \setminus \Delta_2 \) and \( \chi = 0 \) on \( \Delta_2 \setminus \Delta_1 \). Let \( \tilde{h}: \Delta_1 \cup \Delta_2 \rightarrow M \) equal to \( u_2 \) on \( \Delta_2 \setminus \Delta_1 \) and to
\[
f^{-1}_1(\chi f_1 \circ u_1 + (1 - \chi)f_1 \circ u_2) \quad \text{on} \quad \Delta_1.
\]
Then one has
\[ \|u_j - \tilde{h}\|_{W^{1,p}(\Delta_j,M)} < C_1 \varepsilon \quad \text{and} \quad \|\partial_j \tilde{h}\|_{L^p(\Delta_1 \cup \Delta_2,M)} < C_2 \varepsilon, \]
where the positive constants \( C_1, C_2 > 0 \) depend only on \( f_1 \) and \( \chi \). The second inequality means that the origin \( 0 \in W^{1,p}(\Delta_1 \cup \Delta_2, \tilde{h}^*TM) \) is almost a solution of the equation \( F_{\tilde{h}}(\xi) = 0 \). We find a true solution \( \xi \in W^{1,p}(\Delta_1 \cup \Delta_2, \tilde{h}^*TM) \) by using the following version of Proposition A.3.4 from [15].

**Proposition 6.** Let \( X \) and \( Y \) be Banach spaces and \( f \) a \( C^1 \) map taking an open set \( U \subset X \) to \( Y \). For \( x_0 \in U \) let the derivative \( D = d_{x_0} f \) has a bounded right inverse \( Q \). Fix \( c > 0 \) such that \( \|Q\| \leq c \) and \( \delta > 0 \) such that \( x \in U \) and \( \|dx - D\| < \frac{1}{2c} \) for \( \|x - x_0\| < \delta \). If \( \|f(x_0)\| < \frac{\delta}{4c} \) then there exists \( x \in U \) such that \( f(x) = 0 \) and \( \|x - x_0\| < 2c \|f(x_0)\|.

Let \( X = W^{1,p}(\Delta_1 \cup \Delta_2, \tilde{h}^*TM) \), \( Y = L^p(\Delta_1 \cup \Delta_2, \Lambda^{0,1} \otimes \tilde{h}^*TM) \) and \( f = F_{\tilde{h}} \). Fix \( \eta \in L^p(\Delta_1 \cup \Delta_2, \Lambda^{0,1} \otimes \tilde{h}^*TM) \). Let \( \eta_j \) and \( \tilde{h}_j \) be the restrictions of \( \eta \) and \( \tilde{h} \) to \( \Delta_j \). Since \( u_j \) is \( W^{1,p}(\Delta_j) \)-close to \( \tilde{h}_j \), \( D_{\tilde{h}_j} \) admits a bounded right inverse \( Q_{\tilde{h}_j} \). Hence we define a bounded right inverse of \( D_{\tilde{h}} \) by setting \( Q_{\tilde{h}} = Q_{\tilde{h}_j} \eta_j \) on \( \Delta_j \). Let \( \|Q_{\tilde{h}}\| \leq c \). If \( \varepsilon < \delta/(4C_2c) \) one apply the above propostion and take \( h = \exp_{\tilde{h}}(\xi) \) for \( \xi \) satisfying \( F_{\tilde{h}}(\xi) = 0 \).

\[ \square \]

2. APPROXIMATELY ATTACHING A DISC TO A TORUS

We present here a geometric construction similar to the ones in [2, 24]. We use the gluing techniques to avoid the dimension restriction.

**Theorem 7.** Fix \( \varepsilon > 0 \). Let \( u: \overline{D} \to M \) be a J-holomorphic embedding and \( U \) a neighborhood of \( \partial D \). For every \( z \in U \) let \( v_z: \overline{D} \to M \) be a J-holomorphic embedding satisfying \( v_z(0) = u(z) \) and such that the family
\[ G: U \times \overline{D} \to M, \quad G(z, \zeta) = v_z(\zeta) \]
is smooth. There exist a set \( E \subset \partial D \) of measure \( |E| < \varepsilon \) and a J-holomorphic disc \( h: \overline{D} \to M \) such that \( h(0) = u(0) \) and \( \text{dist}(h, G(\partial D \times \partial D)) < \varepsilon \) on \( \partial D \setminus E \).

**Proof.** For \( z \in \partial D \) let \( f_z: \Omega_z \to \mathbb{R}^{2n} \) be a local chart such that \((f_z)_*J = J_{st} \) along \( f_z \circ v_z(\overline{D}) \) and \( \det((f_z)_*J + J_{st}) \neq 0 \) on \( f_z(\Omega_z) \). Further, let \( \Delta_z \) be a disc centered in \( z \) with radius \( r_z > 0 \) such \( \Delta_z \subset U \) and that \( u(\Delta_z) \subset \Omega_z \). We cover \( \partial D \) by \( \Delta_{z_1}, \ldots, \Delta_{z_s} \). Next, we shrink \( r_{z_k} \)'s until \( \Delta_{z_k} \cap \Delta_{z_j} \) becomes empty and \( |E| < \varepsilon \) for \( E = \partial D \setminus \bigcup_{k=1}^s \Delta'_{z_k} \) where \( \Delta'_{z_k} \subset \Delta_{z_k} \). We denote by \( \tilde{D} = D \cup U \) and by \( \Delta_0 = \tilde{D} \setminus \bigcup_{k=1}^s \Delta_{z_k} \).
For $\tau > 0$, consider
\[
W^\tau_k = \{ z \in \mathbb{C}; \text{dist}(z, \partial \mathbb{D} \cap \Delta'_{z_k}) < \tau \text{ and dist}(z, \partial \Delta'_{z_k}) > \tau \}.
\]
Let $\chi_\tau: \tilde{D} \to [0, 1]$ be a smooth function such that $\chi_\tau = 1$ on a slightly smaller subset $\tilde{W}^\tau_k \subset W^\tau_k$, $k = 1, \ldots, s$, and $\chi_\tau = 0$ outside of $\bigcup_{k=1}^s W^\tau_k$.

For $c \in \partial \mathbb{D}$ and $N \in \mathbb{N}$ we define a smooth disc $\phi_{N,c}: \mathbb{D} \to M$ by
\[
\phi_{N,c}(\zeta) = \begin{cases} G(\zeta, \zeta^N \chi_\tau(\zeta)) & \text{for } \zeta \in \bigcup_{k=1}^s W^\tau_k \\ u(\zeta) & \text{otherwise}. \end{cases}
\]

Note that although the derivatives of $\chi_\tau$ go to infinity when $\tau \to 0$, the $\partial J$-derivative of $\phi_{N,c}$ is uniformly bounded due to the $J$-holomorphicity of the fibers $v_z$. Furthermore, since area($W^\tau_k$) tends to zero so does the $L^p$-norm of $\overline{\partial} J \phi_{N,c}$. We want to approximate $\phi_{N,c}$ by a $J$-holomorphic disc.

Let $\phi_k$ be the restriction of $\phi_{N,c}$ to $\Delta_{z_k}$. If we fix $\tau$ small enough the $L^p$-norm of $\overline{\partial} J \phi_k$ will be small. In particular,
\[
d_k = \left\| (w_k)\zeta + A_k(w_k)(\overline{w_k})\zeta \right\|_{L^p(\Delta_{z_k}, \mathbb{R}^{2n})}
\]
will be small, where $w_k = f_{z_k} \circ \phi_k$ and $A_k$ denotes the complex matrix of $J_k = (f_{z_k})_* J$. Hence by Proposition 3, for $\varepsilon_\tau > 0$ there exists $\delta_k > 0$ such that if $d_k < \delta_k$ there exists a $J_k$-holomorphic disc $\tilde{w}_k: \Delta_{z_k} \to \mathbb{R}^{2n}$ such that
\[
\left\| \tilde{w}_k - w_k \right\|_{W^{1,p}(\Delta_{z_k}, \mathbb{R}^{2n})} < \varepsilon_\tau.
\]
Let us assume that $\tau$ is small enough so that $d_k < \min \{ \delta_1, \ldots, \delta_s \}$ for every $k = 1, \ldots, s$. For $\tilde{\phi}_k = f_{z_k}^{-1} \circ \tilde{w}_k$ we have
\[
\left\| \tilde{\phi}_k - \phi_k \right\|_{W^{1,p}(\Delta_{z_k}, M)} < C_1 \varepsilon_\tau,
\]
where $C_1$ depends only on the diffeomorphisms $f_{z_k}$.

Let $\chi: \tilde{D} \to [0, 1]$ be a smooth function such that $\chi = 1$ on $\Delta_0$ and $\chi = 0$ on $\bigcup_{k=1}^s \Delta'_{z_k}$. Let $h: \tilde{D} \to M$ be the smooth map given by
\[
h = \begin{cases} \tilde{\phi}_k & \text{on } \Delta'_{z_k} \\ f_{z_k}^{-1}(\chi w_k + (1 - \chi)\tilde{w}_k) & \text{on } \Delta_{z_k} \setminus \Delta'_{z_k} \\ u & \text{on } \Delta_0. \end{cases}
\]

Note that the $L^p$ norm of $\overline{\partial} J h$ is bounded by $C_2 \varepsilon_\tau$ and that
\[
\left\| h - \tilde{\phi} \right\|_{W^{1,p}(\Delta_{z_k}, M)} < C_3 \varepsilon_\tau,
\]
where the positive constants $C_2$ and $C_3$ depend only on the diffeomorphisms $f_{z_k}$ and on the cut-off $\chi$ (and not on $N \in \mathbb{N}$ or $\tau > 0$).
In order to use Proposition 3, we need to construct a bounded right inverse of the operator $D_{h}$. In the present situation its construction will be more subtle than the one in § 1.2 and follows the lines of [14, 15]. Consider the operator $D_{u}$ restricted to the spaces $X$ and $Y$ from the in Remark 2 for $a_{1} = 0$. We will denote it by $D_{u}^0$. Fix $\eta \in Y$. Denote by $\eta_k$ and by $\tilde{h}_k$ the restrictions of $\eta$ and $h$ to $\Delta_{z_k}$. Since $\tilde{h}_k$ is $W^{1,p}(\Delta_{z_k}, M)$-close to $\tilde{\varphi}_k$, there exists a bounded right inverse $Q_{h_k}$ of $D_{h_k}$. Let also $Q_{u}^0$ be the bounded right inverse of $D_{u}^0$. We set

$$
\xi_0 = Q_{u}^0 \eta \quad \xi_k = Q_{h_k} \eta_k
$$

and we define $\xi = Q\eta$ by

$$
\xi = \begin{cases} 
\xi_k & \text{on } \Delta'_{z_k} \\
\chi \xi_0 + (1 - \chi) \xi_k & \text{on } \Delta_{z_k} \setminus \Delta'_{z_k} \\
\xi_0 & \text{on } \Delta_0.
\end{cases}
$$

It turns out that the map $Q: Y \to X$ is not exactly a right inverse of $D_{h}$. In order to construct a bounded right inverse of $D_{h}$, we need to show the following two estimates:

(9) \[ \|D_{h} \xi - \eta\|_{L^p} \leq \frac{1}{2} \|\eta\|_{L^p}, \]

(10) \[ \|Q\eta\|_{L^p} \leq c \|\eta\|_{L^p} \]

for a positive constant $c > 0$ not depending on $N$ or $\tau$.

Note that on the sets $\Delta'_{z_k}$ and $\Delta_0$, the form $D_{h} \xi - \eta$ vanishes. Moreover, on $\overline{\Delta_{z_k}} \setminus \Delta'_{z_k}$, we have

$$
D_{h} \xi - \eta = D_{h}(\chi \xi_0 + (1 - \chi) \xi_k) - \eta
= D_{h}(\chi(\xi_0 - \xi_k))
= \overline{\partial} \chi \otimes (Q_{u}^0 - Q_{h_k}) \eta + \chi(D_{h_k} Q_{u}^0 - I) \eta
\leq \overline{\partial} \chi \otimes (Q_{u}^0 - Q_{h_k}) \eta + \chi D_{h_k} (Q_{u}^0 - Q_{h_k}) \eta.
$$

Since $\tilde{h}_k$ is $W^{1,p}$-close to $u$ on $\overline{\Delta_{z_k}}$, we obtain

$$
\|D_{h} \xi - \eta\|_{L^p} < C_4 \varepsilon_{\tau} \|\eta\|_{L^p},
$$

where $C_4 > 0$ depends only on $\chi$, $u$ and $f_{z_k}$. Hence we can take $\tau$ small enough so that the estimate (9) is satisfied.

The estimate (10) is satisfied on the sets $\Delta'_{z_k}$ and $\Delta_0$. On $\overline{\Delta_{z_k}} \setminus \Delta'_{z_k}$, we have

$$
\|Q\eta\|_{W^{1,p}} \leq \|\xi_k\|_{W^{1,p}} + \|\chi(\xi_0 - \xi_k)\|_{W^{1,p}}
\leq (C_5 + \|\nabla \chi\|_{L^p}) \|\eta\|_{L^p}.
$$
where \( C_5 > 0 \) depends only on \( \chi, u \) and \( f_{z_k} \). The boundedness of \( \| \nabla \chi \|_{L^p} \) ensures (10).

Due to the estimate (9) we can now define a right inverse of \( D_h \) by setting
\[
Q_h = Q(D_h Q)^{-1}.
\]
Moreover, by the estimate (10) there exist \( c > 0 \) such that \( \| Q_h \| \leq c \) for every \( \tau > 0 \). Since \( \hat{h} \) tends to \( u \) with respect to the \( W^{1,p} \)-topology, \( D_h \) tends to \( D_0^u \) and one can choose an uniform \( \delta > 0 \) in Proposition 6 for small \( \tau \). Hence, there exists \( \xi \in X \) such that \( h = \exp_{\hat{h}}(\xi) \) is the disc we seek. Indeed, it is \( J \)-holomorphic and since we work with \( D_0^u \) one has \( u(0) = h(0) \). This finishes the proof.

\[ \square \]

**Remark 3.** By Proposition 3 we can fix points \( \zeta_{j,l}, l = 1, \ldots, l_j \) in \( \bigcup_{k=1}^{s} \hat{W}_k \) so that \( \hat{h}(\zeta_{j,l}) = G(\zeta_{j,l}, c_{\zeta_{j,l}}^N) \). Moreover, if we restrict the operator \( D_u \) as in the Remark 2 we may assume that \( h(\zeta_{j,l}) = G(\zeta_{j,l}, c_{\zeta_{j,l}}^N) \).

**3. Proof of Theorem 1 and Theorem 2**

Recall that an upper semi-continuous function \( f \) defined on an open set \( V \) in \( (M, J) \) is \( J \)-plurisubharmonic if \( f \circ u \) is subharmonic for any \( J \)-holomorphic disc \( u : \Delta \rightarrow V \). We denote by \( \text{Psh}_J(V) \) the set of \( J \)-plurisubharmonic functions on \( V \).

The present proof of Theorem 1 follows the one in [22] but replaces the Cartan lemma by gluing techniques. This new approach enables us to avoid dimensional restrictions such as the ones in [10] (and [7] for Theorem 2). However, the reader should note that a large part of the proofs in this last two references goes trough in any dimension. Hence, we drop some details. In particular, we do not prove that \( EP_f \) is upper semi-continuous in higher dimensions. This should be obvious by reading the page 267 in [10].

**Proof of Theorem 1.** Consider \( EP_f \) for some upper semi-continuous function \( f \) defined on \( M \). Since \( EP_f \) is upper semi-continuous it can be approximated from above by continuous functions. Hence it is enough to assume that \( f \) is continuous (see [5, p. 8]). We need to show that, given \( p \in M \) with \( EP_f(p) > -\infty \) and for every small smooth \( J \)-holomorphic embedding \( u_p \) centered at \( p \), we have
\[
\text{EP}_f(p) = \text{EP}_f(u_p(0)) \leq \int_0^{2\pi} EP_f \circ u_p(e^{i\theta}) \, d\theta / 2\pi.
\]

(11)
Fix such a disc $u_p$ and a number $\varepsilon > 0$. Let $\mathbb{D}_{u_p}$ denote a small neighborhood of $\mathbb{D}$ on which $u_p$ is defined, and let $z \in \mathbb{D}_{u_p}$. There exists a $J$-holomorphic disc $v_z$ centered at $u_p(z)$ such that

\begin{equation}
\int_0^{2\pi} f \circ v_z(e^{it}) \frac{dt}{2\pi} < EP\{u_p(z)\} + \frac{\varepsilon}{2\pi}.
\end{equation}

Since $\dim_{\mathbb{R}} M \geq 6$, we may assume that this disc $v_z$ is embedded [26].

If a point $z'$ is close enough to $z$ one can perturb $v_z$ into an embedded disc $v_z^\ast$ centered at $u_p(z')$ and satisfying (12), see [10, Corollary 4]. Let us cover $\partial \mathbb{D}$ by a finite number of sets

$$U_z = \{ z' \in \mathbb{D}_{u_p}; v_z^\ast \text{ is well defined} \}.$$ 

Denote them by $U_{z_1}, U_{z_2}, \ldots, U_{z_m}$ and let $U_{z_{m+1}} = U_{z_1}$.

We may assume that the triple intersections are empty and that $U_{z_j}$ meets precisely the sets $U_{z_{j-1}}$ and $U_{z_{j+1}}$. We pick a point $p_j \in U_{z_j} \cap U_{z_{j+1}}$. On a small neighborhood of $u_p(p_j)$, the set of small $J$-holomorphic discs is in bijective correspondence with the set of small standard holomorphic discs in $\mathbb{C}^n$. Let $V_j$ be the set of all $z' \in \mathbb{D}_{u_p}$ for which $u_p(z')$ lies in this neighborhood. We shrink the sets $U_{z_j}$ until they become pairwise disjoint but keep them large enough so that together with the sets $V_j$ they still cover $\partial \mathbb{D}$ and that the measure of $E_1 = \partial \mathbb{D} \setminus \bigcup_{j=1}^m U_{z_j}$ is small.

We define a smooth function $\chi_j$ on $U_{z_j}$ such that it equals to 1 on a slightly smaller closed subset of $U_{z_j} \setminus (V_j \cup V_{j-1})$ and that it has a small positive value on a slightly smaller closed subset of $U_{z_j} \cap V_j$ and $U_{z_j} \cap V_{j-1}$. We redefine the discs $u_j^\ast$ into the maps $\zeta \mapsto u_j^\ast(\chi_j(z'))$. The discs with centers in $u(U_{z_j} \cap V_j)$ and $u(U_{z_j} \cap V_{j-1})$ are small embeddings. We apply the bijective correspondence defined in the neighborhood of $u_p(p_j)$ to construct discs with centers in $u(V_j \setminus (U_{z_j} \cup U_{z_{j+1}}))$.

Let us denote now by $G$ the constructed family of embeddings defined on $\bigcup_{j=1}^m (U_{z_j} \cup V_j) \times \mathbb{D}$. Since the measure of $E_1$ is small and almost every disc satisfies (12), we have the following crucial inequality:

\begin{equation}
\int_0^{2\pi} \int_0^{2\pi} f \circ G(e^{i\theta}, e^{it}) \frac{d\theta}{2\pi} \frac{dt}{2\pi} < \int_0^{2\pi} f \circ u_p(e^{i\theta}) \frac{d\theta}{2\pi} + \varepsilon.
\end{equation}

Let $h$ be a $J$-holomorphic disc centered at $p$ and with the boundary approximately attached to $\Lambda = G(\partial \mathbb{D} \times \partial \mathbb{D})$ given by Theorem 7. By construction in the proof of Theorem 7 we have

$$\text{dist}(h(z), G(z, cz^N)) < \varepsilon_2$$
for $c \in \partial \mathbb{D}$, $N \in \mathbb{N}$ large and $z \in \partial \mathbb{D} \setminus E_2$ where $|E_2| < \varepsilon_2$. Hence, if $\varepsilon_1, \varepsilon_2 > 0$ are small
\[
\int_0^{2\pi} f \circ h(e^{it}) \frac{d\theta}{2\pi} < \int_0^{2\pi} f \circ G(e^{it}, e^{i(t+N\theta)}) \frac{d\theta}{2\pi} + \varepsilon.
\]
Here $c = e^{it}$. We set
\[
I_t = \int_0^{2\pi} f \circ G(e^{it}, e^{i(t+N\theta)}) \frac{d\theta}{2\pi}, \quad t \in [0, 2\pi]
\]
By the mean value theorem there exists a value $\nu \in [0, 2\pi)$ such that
\[
I_{\nu} = \int_0^{2\pi} f \circ h(e^{it}) \frac{d\theta}{2\pi} \leq I_t + \varepsilon \leq \int_0^{2\pi} f \circ G(e^{it}, e^{it}) \frac{d\theta}{2\pi} + 2\varepsilon.
\]
Together with (13) we obtain
\[
EP_f(u_p(0)) \leq \int_0^{2\pi} f \circ h(e^{it}) \frac{d\theta}{2\pi} \leq I_{\nu} + \varepsilon \leq \int_0^{2\pi} EP_f \circ u_p(e^{it}) \frac{d\theta}{2\pi} + 2\varepsilon.
\]
Since $\varepsilon > 0$ was arbitrary, (11) holds and the theorem is proved. \(\square\)

Now we turn to the proof of Theorem 2. First, we extract some facts from the pages 8 and 9 in [7]. Given a nonnegative function $\alpha$ on $M$ let
\[
F_{\alpha} = \{ f \in \text{Psh}_J(M); f \leq 0, \nu f \geq \alpha \}.
\]
If $f \in F_{\alpha}$ and $u \in \mathcal{O}_J(D, M)$, then $f \circ u \leq 0$ is a subharmonic function in a neighborhood of $D$ whose Lelong number at any point $\zeta \in D$ satisfies
\[
\nu f \circ u(\zeta) \geq \alpha(\nu u(\zeta)).
\]
Hence $f \circ u$ is bounded above by the largest subharmonic function $v_{\alpha \circ u} \leq 0$ on $D$ satisfying $\nu v_{\alpha \circ u} \geq \alpha \circ u$. This maximal function $v_{\alpha \circ u}$ is the weighted sum of Green functions with coefficients $\alpha \circ u$:
\[
v_{\alpha \circ u}(z) = \sum_{\zeta \in D} \alpha(\nu u(\zeta)) \log \left| \frac{z - \zeta}{1 - \zeta \bar{z}} \right|, \quad z \in D.
\]
(If the sum is divergent then $v \equiv -\infty$.)

Setting $z = 0$, we see that for every $f \in F_{\alpha}$ and $u \in \mathcal{O}_J(D, M)$ we have
\[
f(u(0)) \leq \sum_{\zeta \in D} \alpha(\nu u(\zeta)) \log |\zeta| \leq \inf_{\zeta \in D} \alpha(\nu u(\zeta)) \log |\zeta| =: K_\alpha(u).
\]
By taking infima over all $J$-holomorphic discs $u$ centered at $p \in M$ we obtain for any $f \in F_{\alpha}$:
\[
f(p) \leq E\alpha(p) \leq EK_\alpha(p) =: k_\alpha(p).
\]
Next, note that [7, Lemmas 2 and 4] has no dimensional restrictions. Hence, proving our Theorem 1 we also get [7, Lemma 8], that is, $EP_{k_{\alpha}} = v_{\alpha}$.

**Proof of Theorem 2.** By taking supremum over all $f \in \mathcal{F}_{\alpha}$ in (15) we get $EP_{k_{\alpha}} = v_{\alpha} \leq EL_{\alpha}$, therefore it suffices to prove that $EL_{\alpha} \leq EP_{k_{\alpha}}$. Equivalently, we need to show that for every continuous function $\varphi : M \rightarrow \mathbb{R}$ with $\varphi \geq k_{\alpha}$, embedded $J$-holomorphic disc $u : \overline{\mathbb{D}} \rightarrow M$, and $\epsilon > 0$ there exists a $J$-holomorphic disc $h$ such that $h(0) = u(0)$ and

$$L_{\alpha}(h) = \sum_{z \in \mathbb{D}} \alpha(h(z)) \log |z| < \frac{1}{2\pi} \int_{0}^{2\pi} \varphi \circ u(e^{it})dt + \epsilon.$$ 

Let $\mathbb{D}_{u}$ denote a small neighborhood of $\overline{\mathbb{D}}$ on which $u$ is defined, and let $z \in \mathbb{D}_{u}$. Since $k_{\alpha} = EK_{\alpha}$ there exists a $J$-holomorphic disc $v_{z}$ and a point $b \in \mathbb{D}^{*}$ such that $v_{z}(0) = u(z)$ and

$$(16) \quad \alpha(v_{z}(b)) \log |b| < \varphi(u(z)) + \frac{\epsilon}{2}.$$ 

Again, we may assume that $v_{z}$ is embedded [26]. Using [7, Lemma 4] we can perturb $v_{z}$ into $v_{z}^{*}$ centered at $u(z)$ satisfying (16) such that $v_{z}^{*}(b) = v_{z}(b)$. Hence, by procedure similar to the one in the proof of Theorem 1 we obtain sets $U_{j}$ and $V_{j}$ such that a family $G : \bigcup_{j=1}^{m} (U_{j} \cup V_{j}) \times \overline{\mathbb{D}} \rightarrow M$ has the following two properties: $G(z,0) = u(z)$ and

$$(17) \quad \alpha(y_{j}) \log |b_{j}| \cdot |I| < \int_{I} \varphi \circ u(e^{it})\frac{dt}{2\pi} + \frac{\epsilon}{2}.$$ 

Here $|I|$ denotes the normalized arc length of an arbitrary arc $I \subset U_{j} \cap \partial \mathbb{D}$ and $b_{j} \in \mathbb{C}^{*}$ such that $G(z,b_{j}) = v_{z}(b_{j}) = y_{j}$ for every $z \in U_{j}$.

Let $h$ be a $J$-holomorphic disc centered at $p$ and with the boundary approximately attached to $G(\partial \mathbb{D} \times \partial \mathbb{D})$ from Theorem 7. By construction the points $h(\zeta)$ are close to $G(\zeta,c\zeta^{N})$ for $\zeta \in \hat{W}_{\tau}^{k}$. Note that $\tau > 0$ has to be small enough, but $N \in \mathbb{N}$ and $c \in \partial \mathbb{D}$ may be chosen arbitrarily. We set $c = 1$ and $I_{j}^{k} \subset \subset U_{j} \cap \hat{W}_{\tau}^{k} \cap \partial \mathbb{D}$ (most of them are empty) such that

$$\left| \int_{\partial \mathbb{D} \cup \bigcup_{j,k=1}^{m} I_{j}^{k}} \varphi \circ u(e^{it})\frac{dt}{2\pi} \right| < \frac{\epsilon}{2}. \tag{18}$$

(This is possible since the measure of $\bigcup_{j=1}^{m} (V_{j} \cap \partial \mathbb{D})$ and $\partial \mathbb{D} \cup \bigcup_{k=1}^{m} W_{\tau}^{k}$ are small.) By [7, Lemma 7] there exists $N_{0} \in \mathbb{N}$ such that the equation $\zeta^{N} = b_{j}$ has at least $N|I_{j}^{k}|$ solutions in every $\hat{W}_{\tau}^{k} \cap \mathbb{D}$ for $N \geq N_{0}$. Let us denote by $\zeta_{j,l}$, $l = 1,\ldots,l_{j}$, the solutions lying in $U_{j}$. Note that $l_{j} > N \sum_{k=1}^{m} |I_{j}^{k}|$. 

\textbf{Figures and Tables:}

1. \textbf{Figure 1:} Illustration of the disc functional $D_{k}$.
2. \textbf{Table 1:} Comparison of disc functionals $D_{k}$ and $D_{k_{\alpha}}$.

\textbf{References:}

By Remark 3 we may assume that
\[ h(\zeta_{j,l}) = G(\zeta_{j,l}, \zeta_{j,t}^n) = G(\zeta_{j,l}, b_j) = y_j. \]
We now have
\[
\sum_{\zeta \in \mathcal{D}} \alpha(h(\zeta)) \log |\zeta| \leq \sum_{j=1}^m \alpha(y_j) \sum_{l=1}^{l_j} \log |\zeta_{j,l}|.
\]
Furthermore, note that
\[
\sum_{l=1}^{l_j} \log |\zeta_{j,l}| = \frac{1}{n} \sum_{l=1}^{l_j} \log |b_j| \leq \sum_{k=1}^s |I_j^k| \log |b_j|,
\]
Combining this with (17) and (18) we get
\[
\sum_{\zeta \in \mathcal{D}} \alpha(h(\zeta)) \log |\zeta| < \sum_{j=1}^m \alpha(y_j) \sum_{k=1}^s |I_j^k| \log |b_j| < \int_0^{2\pi} \varphi \circ u(e^{it}) \frac{dt}{2\pi} + \epsilon.
\]
The proof is complete. \[ \square \]

4. Applications

4.1. Regularization of $J$-plurisubharmonic functions. Recently S. Pliš [20] proved that $J$-plurisubharmonic functions can be locally approximated by smooth $J$-plurisubharmonic functions in real dimension four. Along with the almost complex Richberg theorem S. Pliš obtained in [19], the four dimensional version of Theorem 1 (see [10]) is the key ingredient in [20]. As a direct consequence of Theorem 1 and [19, 20], we obtain

**Corollary 8.** Let $(M, J)$ be an almost complex manifold and let $p \in M$. There exists a neighborhood $V$ of $p$ such that for every $J$-plurisubharmonic function $\rho$ on $V$ there exists a decreasing sequence of $J$-plurisubharmonic $C^\infty$ functions $\rho_k$ on $V$ such that $\rho_k \to \rho$.

Very recently, Harvey, Lawson and Pliš [8] proved the corresponding global approximation result in case $M$ is $J$-pseudoconvex (see Theorem 4.1 in [8]), i.e. when $M$ admits a smooth strictly $J$-plurisubharmonic exhaustion function. In particular, they also obtain Corollary 8 (see Corollary 4.2) since any almost complex manifolds admits a system of (strictly) $J$-pseudoconvex neighborhoods. It is worth mentioning that their techniques are very different from ours and that in our case, the local approximation is purely an application of our main Theorem 1.
4.2. Psh$_J$-hull and Poletsky discs. In addition to providing new constructions of $J$-plurisubharmonic functions by variational methods, Poletsky theory also gives beautiful characterizations of hulls in terms of $J$-holomorphic discs. The most classical application of Theorem 1 in case $(M,J) = (\mathbb{C}^n,J_{st})$ is certainly the characterization of the polynomial hull of compact subsets of $\mathbb{C}^n$ by holomorphic discs (see [18]).

In our case, we obtain a characterization of compact Psh$_J$-hulls of compact sets in an almost complex manifold in terms of $J$-holomorphic discs similar to (see [11, 21, 5] and also [13]). Let $K$ be a compact subset of an almost complex manifold. We define the Psh$_J$-hull $\hat{K}$ of $K$:

$$\hat{K} = \{ p \in M; \rho(p) \leq \sup_K \rho \ \forall \rho \in \text{Psh}_J(M) \}$$

Note that due to the recent result of Harvey, Lawson and Plíš [8], in case the manifold $M$ is $J$-pseudoconvex the Psh$_J$-hull previously defined coincides with the Psh$_J$-hull with respect to continuous or smooth $J$-plurisubharmonic functions (see Corollary 6.3 in [8]).

**Corollary 9.** Let $K$ be a compact in an almost complex manifold $(M,J)$ with a compact Psh$_J$-hull $\hat{K}$. Let $V$ be a relatively compact open set containing $\hat{K}$. Then a point $p \in M$ belongs to $\hat{K}$ if and only if for any $\varepsilon > 0$ and for any neighborhood $U \subset V$ of $K$, there exists a $J$-holomorphic disc $u : \Delta \to M$ centered at $p$ and a set $E \subset \partial \Delta$ of measure $|E| < \varepsilon$ such that $u(\partial \Delta \setminus E) \subset U$.

We repeat the classical arguments of [5] for instance.

**Proof.** Let $\varepsilon > 0$ and let $U$ be a neighborhood of $K$ such that $U \subset V$. Assume $p \in \hat{K}$. We define an upper semi-continuous function $f$ by setting $0$ in $U$ and by $1$ in $V \setminus U$. According to Theorem 1, the function $\hat{f}$ is $J$-plurisubharmonic on $M$. Then $\hat{f}(p) = 0$ since $\hat{f}$ is equal to $0$ on $K \subset U$ and $p \in \hat{K}$. Therefore, there exists a $J$-holomorphic disc $u : \Delta \to M$ with $u(0) = p$ and

$$|E| = \int_0^{2\pi} f \circ u(e^{i\theta}) \frac{d\theta}{2\pi} < \varepsilon,$$

where $E = \{ \zeta \in \partial \Delta; u(\zeta) \notin U \}$.

Reciprocally, let $p \in M$ be such that for any $\varepsilon > 0$ and any neighborhood $U \subset V$ of $K$, there exists a $J$-holomorphic disc $u$ centered at $p$ and a set $E \subset \partial \Delta$ of measure $|E| < \varepsilon$ such that $u(\partial \Delta \setminus E) \subset U$. Let $\rho$ be a $J$-plurisubharmonic function. Let $\varepsilon > 0$ and $U \subset V$ containing $K$ such that...
sup_U \rho \leq \sup_K \rho + \varepsilon. \ Let u be a J-holomorphic disc and E \subset \partial \Delta be a set as in the statement of Corollary 9. We have
\[
\rho(p) \leq \int_0^{2\pi} \rho \circ u(e^{i\theta}) \frac{d\theta}{2\pi} = \int_E \rho \circ u(e^{i\theta}) \frac{d\theta}{2\pi} + \int_{\partial \Delta \setminus E} \rho \circ u(e^{i\theta}) \frac{d\theta}{2\pi} \leq \varepsilon \sup_V \rho + \sup_K \rho + \varepsilon.
\]
It follows that \(\rho(p) \leq \sup \rho\) and thus \(p \in \hat{K}\).

\[
\Box
\]

**Remark 4.** Note that in case \(M\) admits a global strictly J-plurisubharmonic function, the compactness assumption made on the Psh\(_J\)-hull \(\hat{K}\) in Corollary 9 may be replaced by any of the equivalent conditions in Theorem 5.4 of Diederich and Sukhov [3]; e.g. one obtains a characterization of Psh\(_J\)-hulls of compact sets contained in Stein domains.

Another application of Theorem 1 is related to finding J-holomorphic discs with prescribed center and with most of their boundaries in a given open set. This property was pointed out by Larusson and Sigurdsson in [11], (see also [22]). We omit the proof.

**Corollary 10.** Let \((M, J)\) be an almost complex manifold admitting no non-constant bounded J-plurisubharmonic function. Let \(V \subset M\) be a nonempty open set and let \(p \in M\). Then for every \(\varepsilon > 0\), there exists a J-holomorphic disc \(u : \Delta \to M\) centered at \(p\) and a set \(E \subset \partial \Delta\) of measure \(|E| < \varepsilon\) such that \(u(\partial \Delta \setminus E) \subset V\).

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REFERENCES


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