
Reviewed by Steven R. Bell

I used to like to downhill ski hard and fast, and so I always ended up “single” in the lines for the chair lift. I’d yell “Single!” and someone, perhaps near the front of the line, would yell “Here,” and in no time, I’d find myself zinging up the hill sitting next to a total stranger.

Skiing puts most people in a good mood, and if the wind wasn’t too cold, we’d start up a conversation, which invariably went . . .

“Where’ya from?”
“Princeton, New Jersey.”
“Wha’dyu do there?”
“Teach math.”
“Math? At the University?”
“Yeah.”

Sometimes, the reaction to this statement was similar to what I would have gotten if I had said that I was a mortician from Hoboken. But more often, the answer led to a long onslaught of questions about the nature of mathematical research. What is it? How do you do it? Why do you do it? Mercifully, chair lifts in the east are short enough that the conversation would come to an abrupt halt at about the time my partner started to look at me with quizzical, upraised eyebrows, especially after I mentioned my passion for biholomorphic mappings between weakly pseudoconvex domains.

During the years I was a postdoc and instructor at Princeton, I never felt that I gave satisfactory answers to these questions. (And later, Brooke Shields became a student at Princeton, which meant that the conversations on the chair lifts usually took a whole different turn.) Now, when I have a rare burst of energy to ski single, I ask my curious chairlift partners if they’ve seen the wonderful PBS special, starring Andrew Wiles, about the solution to Fermat’s last theorem. If they haven’t, I can quote Wiles’s superb description of the nature of mathematical research and the sheer bliss that follows success.

Today, if I were to share a chair with an undergraduate math major who seemed ready for a chance to grope around alone in a dark room of mathematics, I might also mention this little volume by John D’Angelo. D’Angelo, flashlight in hand, guides his readers through the dimly lit but hallowed halls of complex analysis, through the servants’ quarters and up a secret stairway in the back that leads to the attic. Once in the attic, he opens a dusty trunk and pulls out a new insight into Hilbert’s seventeenth problem based on his own research into proper holomorphic mappings between complex balls of different dimensions and the Bergman kernel function.
I delight in telling stories about mathematical research such as D’Angelo spins here. Why should simple inequalities in complex variables lead to proper holomorphic mappings in several variables, and to Hilbert space, and to the Bergman kernel function, and to Hilbert’s seventeenth? Why does groping around in the mathematical darkness so often lead to such lovely strings of ideas?

Let me now spin D’Angelo’s tale of mathematical discovery in my own words. Every course in complex variables spends a day or two on the famous Möbius transformations,

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$ 

When \(a\) is a fixed complex number in the unit disc in the plane, these analytic functions of \(z\) map the unit disc one-to-one onto itself, and they do it conformally. It is often left as an exercise to show that these maps really are one-to-one and onto. It is a tricky little exercise using inequalities to show that \(|\varphi_a(z)| < 1\) when \(|z| < 1\) and \(|\varphi_a(z)| = 1\) when \(|z| = 1\). Next, ask yourself if there are other rational functions \(R(z)\) that satisfy \(|R(z)| < 1\) when \(|z| < 1\) and \(|R(z)| = 1\) when \(|z| = 1\). This is also sometimes given as an exercise. You can use the maximum principle and Möbius transformations to show that any such rational function \(R\) is given by a finite product of Möbius transformations,

$$R(z) = \lambda \prod_{n=1}^{N} \left( \frac{z - a_n}{1 - \bar{a}_n z} \right)^{m_n},$$

where \(\lambda\) is a unimodular constant, the \(a_n\) are points in the unit disc, and the \(m_n\) are positive integers. Such rational functions are called finite Blaschke products, and they arise many places in complex analysis. To continue in this direction, it would be natural to ask: What are the analytic functions \(f\) on the unit disc that satisfy \(|f(z)| < 1\) when \(|z| < 1\) and \(|f(z)| \to 1\) as \(|z| \to 1\)? Such functions are known as proper holomorphic self-maps of the unit disc. Another, more topological, way to describe them is to say that \(f\) is analytic and \(f^{-1}(K)\) is compact whenever \(K\) is a compact subset of the disc. It is remarkable that a minor refinement of the argument in the rational case shows that the proper holomorphic self-mappings of the unit disc are precisely the finite Blaschke products.

These results about self-mappings of the disc can be proved in many ways using standard tools like the Schwarz lemma, the maximum modulus principle, or even the Schwarz reflection principle, that are second nature to every complex analyst. However, when similar questions are asked about mappings between unit balls in higher dimensional complex Euclidean spaces, the beloved standard tools of one complex variable go out the window, and that’s where Hilbert space, the Bergman kernel, and D’Angelo’s clever applications of linear algebra enter the picture.

The Cauchy-Riemann equations are derived in the first few days of any course on complex variables. If \(f(x + iy) = u(x, y) + iv(x, y)\) is analytic, then

$$f' = u_x + iv_x = v_y - iu_y.$$ 

Now, if we view the analytic function \(f\) as a mapping from \(\mathbb{R}^2\) to itself via \((x, y) \mapsto (u(x, y), v(x, y))\), it follows from the Cauchy-Riemann equations that the Jacobian of the map is given by

$$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det \begin{bmatrix} u_x & -v_x \\ v_x & u_y \end{bmatrix} = |f'|^2.$$
This last observation suggests that a Hilbert space is lurking nearby. Indeed, if \( f \) is a one-to-one holomorphic mapping of a domain \( \Omega_1 \) onto a domain \( \Omega_2 \), the change of variables formula for transformations of \( \mathbb{R}^2 \) yields that

\[
\iint_{\Omega_1} |f'|^2 (\varphi \circ f) \, dA = \iint_{\Omega_2} \varphi \, dA,
\]

and it follows that the mapping \( \varphi \mapsto f' \cdot (\varphi \circ f) \) is an isometry between \( L^2(\Omega_2) \), the space of complex-valued functions on \( \Omega_2 \) that are square integrable with respect to Lebesgue area measure \( dA \), and \( L^2(\Omega_1) \). For a given domain \( \Omega \), the square integrable analytic functions form a lovely closed subspace of \( L^2(\Omega) \) known as the Bergman space \( B^2(\Omega) \). The Bergman kernel function is the kernel for the projection \( P \) of \( L^2(\Omega) \) onto \( B^2(\Omega) \). For the unit disc, the formula is

\[
(P\varphi)(z) = \iint_{w \in D_1(0)} K(z, w) \varphi(w) \, dA,
\]

where

\[
K(z, w) = \frac{1}{\pi (1 - \overline{z}w)^2}.
\]

D’Angelo explains these things, as well as the fact that if \( f \) is a conformal mapping between \( \Omega_1 \) and \( \Omega_2 \), then the Bergman kernel function transforms according to

\[
K_{\Omega_1}(z, w) = f'(z) K_{\Omega_2}(f(z), f(w)) f'(w).
\]

Now here’s a silly idea. Suppose that \( f \) is a conformal mapping of the unit disc onto itself that fixes the origin. The Schwarz lemma spits out instantly that \( f(z) = \lambda z \) for some unimodular constant \( \lambda \). However, if we were prohibited from using the Schwarz lemma, we might put our conformal mapping into the transformation formula for the Bergman kernel, taking \( w = 0 \), to obtain

\[
1 = f'(z) f'(0).
\]

This also shows that \( f(z) = \lambda z \), where \( \lambda = f'(0) \) is a unimodular constant. So the idea works! And it is not nearly so silly in several variables as it is in one. If we had assumed that \( f(a) = 0 \), then the same argument, taking \( w = a \), would yield that

\[
\frac{1}{(1 - \overline{a}z)^2} = f'(z) f'(a),
\]

and it follows rather easily that \( f \) must be a unimodular constant times a Möbius transformation. Thus, we might start up those back stairs to the attic.

The unit ball in complex Euclidean space \( \mathbb{C}^n \) is the set of complex vectors \((z_1, z_2, \ldots, z_n)\) such that \( \|z\| < 1 \), where \( \|z\|^2 = \sum_{j=1}^{n} |z_j|^2 \). The Bergman kernel can be used as before to determine the one-to-one analytic self-mappings of the unit ball in \( \mathbb{C}^n \). They are rational functions that look very much like the Möbius transformations. Now brace yourself for a surprise. The proper holomorphic self-mappings of the unit ball are given by exactly the same maps, i.e., every proper holomorphic self-map of the unit ball in \( \mathbb{C}^n \) when \( n > 1 \) is one-to-one and onto. This is a fascinating result that requires a bit more machinery to attack than its one-variable analogue. I like
to say that this result shows that holomorphic mappings of two variables are more than twice as interesting as mappings of one variable (and I have a colleague who always counters that, no, this shows that they are less than half as interesting).

After this shock, it is imperative to ask more questions. What are the polynomial maps $Q$ from $\mathbb{C}^n$ to $\mathbb{C}^m$ such that $\|Q(z)\| < 1$ when $\|z\| < 1$ and $\|Q(z)\| = 1$ when $\|z\| = 1$? What are the rational maps $R$ such that $\|R(z)\| < 1$ when $\|z\| < 1$ and $\|R(z)\| = 1$ when $\|z\| = 1$? What are the proper holomorphic maps between balls of different dimensions? The answers to these questions lead to many more questions, and the answers are so intriguing that I won’t give them away here.

D’Angelo became the world’s expert in dealing with hypersurfaces like the boundary of the unit ball in several variables while studying the regularity of solutions of the Cauchy-Riemann equations in many variables. The tools he developed to describe certain hypersurfaces as graphs of defining functions given as a sum of squares minus another sum of squares started him on a path that would lead to his insights into the mapping problems discussed earlier and a new take on Hilbert’s seventeenth problem about decomposing positive rational functions as sums of squares. It is lovely mathematics and D’Angelo guides his readers not along the path he took himself but along an efficient path that he has blazed after many years of reflection on the subject.

To be a creative mathematician, you need to think differently from the herd of mathematicians around you. I especially enjoyed D’Angelo’s book because I got a glimpse into his highly individual mode of thought. Most textbooks on complex analysis, for example, remark how it is possible to construct harmonic conjugates of harmonic polynomials in a formal way by viewing the variables $z$ and $\bar{z}$ as independent. It is a moment in mathematics that one faces with a sense of beauty and mystery. And the moment is fleeting, because most complex analysis books get right back on track and start piling theorem upon lemma with absolute rigor without ever coming back to this tantalizing subject. But D’Angelo dwells on it and savors it and turns it into one of the main themes of the book.

D’Angelo’s book is delightful, but I must warn a casual reader that the experience will be like a game of “crack the whip” at the ice skating pond. D’Angelo starts out so slowly and gently that an undergraduate reader might be able to race through the opening pages of the book. When a reader becomes aware of the sizable acceleration, it is probably time to go back a few pages and really sit down and do those exercises that seemed so inconsequential when D’Angelo casually tossed them out. I’m not sure many undergraduates could hold onto the rope until the end of the game, but I am sure that whenever they let go, they will be jettisoned with a good deal of momentum that could send them into any number of avenues of study and research. A graduate student could hold on longer and benefit even more. I held on to the end, but my gloves are still smoking.