Representations of $SL(2) \times G$

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This paper is dedicated to my teacher, Goro Shimura.

Abstract. Let $\mathfrak{g}$ be a complex reductive Lie algebra. We investigate a formalism for producing a natural model for representations of $\mathfrak{sl}(2) \times \mathfrak{g}$ which decompose in such a way so as to produce interesting correspondences between representations of $\mathfrak{sl}(2)$ and of $\mathfrak{g}$. Our formalism generalizes the standard dual pair $SL(2) \times O(2n)$. We use it to obtain a formally unitary representation of $\mathfrak{sl}(2, \mathbf{R}) \times \mathfrak{u}(3)$ that essentially realizes the symmetric square lift of Gelbart and Jacquet. We lift this representation of Lie algebras to a unitary representation of $SL(2, \mathbf{R}) \times U(3)$ on a certain L^2 space, with quite explicit formulas.

1. Introduction

We are motivated by the problem of seeking a natural realization of Gelbart and Jacquet's symmetric square lift of automorphic forms (or automorphic representations) on $GL(2)$ to automorphic forms on $GL(3)$ [**GJ**] in terms of something akin to, but not quite identical with, a theta-correspondence. This would lead to an integral kernel for the lifting, and perhaps even some insights into whether one could realize some other cases of Langlands functoriality similarly. Actually, it is more convenient to consider a variant of the symmetric square lift, namely the adjoint lift, corresponding to the map on the dual groups $PGL(2) \to SL(3)$; so the goal is to relate forms on $SL(2)$ to forms on $PGL(3)$, or rather forms on $SL(2)$ to forms on $GL(3)$ with trivial central character. A first step in this direction is to look for reasonable unitary representations V of $SL(2, F) \times GL(3, F)$, F a local field, with properties similar to local Howe duality (see section 6 of [**Ho**]): i.e., the irreducible quotients of V should have the form $V_2 \otimes V_3$, where V_2 is an irreducible representation of $SL(2, F)$ and V_3 is the corresponding representation of $GL(3, F)$ under the adjoint lift. The usual construction of V by a theta-correspondence would involve taking V to be the minimal representation of a big reductive group G containing $SL(2) \times GL(3)$ as a dual pair. Unfortunately those few cases in which groups of type $A_1 \times A_2$ arise as dual pairs seem to yield less interesting correspondences. We

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must therefore look for a representation V of $SL(2) \times GL(3)$ in the absence of such a big group.

Our results in this paper are all for $F = \mathbf{R}$, but we begin by describing a technique that should work for arbitrary F. Our aim in this introduction is to first explain some general ideas which motivated the construction in this paper, without worrying about certain technical details. We then summarize the contents of Sections 2–4, which contain the actual results of this paper. Those results are motivated by the considerations discussed in this introduction, but end up taking a somewhat different form.

We first describe the effect of the adjoint lift on "most" principal series representations. Let $\chi : F^\times \to \mathbf{C}^\times$ be a multiplicative character. Let PS_2^χ be the unitarily induced principal series representation of $SL(2,F)$, obtained by unitarily inducing the character $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right) \mapsto \chi(a)$ from the Borel subgroup B of $SL(2,F)$ up to the whole group. (In fact, even if χ is not unitary, $PS_2^\chi \cong PS_2^{\chi^{-1}}$ for "generic" χ.) According to the local adjoint lift, PS_2^χ corresponds to the representation PS_3^χ of $GL(3,F)$ obtained by unitarily inducing the character $\chi \circ \lambda$ of its Borel subgroup up to the whole group, where λ is the following rational character of the Borel subgroup:

$$\lambda(\begin{pmatrix} a & * & * \\ & b & * \\ & & c \end{pmatrix}) = a/c. \tag{1.1}$$

One can also view λ as an integral dominant weight of $GL(3)$. We note that $PS_3^\chi \cong PS_3^{\chi^{-1}}$ for generic χ, just as for PS_2^χ.

Write $N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right) \mid b \in F\}$ for the unipotent radical of the Borel subgroup B. Given a representation W of $SL(2)$, write ${}^N W$ for its Jacquet module of N-coinvariants; ${}^N W$ carries a representation of the multiplicative group $F^\times$, identified with the maximal torus in B. PS_2^χ is characterized by the fact that ${}^N PS_2^\chi$ is two-dimensional, and that $\mathbf{G}_m$ essentially acts on it by $\chi \oplus \chi^{-1}$. To be precise, this is slightly inaccurate, due to the presence of normalizing factors from the unitary induction up to $SL(2)$. These factors are thankfully absorbed into the normalizing factors of the induction up to $SL(3)$ and shall henceforth be ignored.

Our desired representation V of $SL(2) \times GL(3)$ must therefore "mostly" satisfy

$${}^N V \cong \int_{\chi : F^\times \to \mathbf{C}^\times} \chi^{-1} \otimes PS_3^\chi, \qquad \text{as an } F^\times \times GL(3)\text{-representation.} \tag{1.2}$$

(By "mostly" we mean that the subspace of ${}^N V$ transforming under $F^\times$ by χ need only be isomorphic to PS_3^χ for generic χ.) On the other hand, there is a rather natural space on which $F^\times \times GL(3)$ acts by $\int_\chi \chi^{-1} \otimes PS_3^\chi$. It is the space $\mathcal{S}(\mathcal{O}_\lambda)$ of Schwartz class functions on the orbit $\mathcal{O}_\lambda$ of a highest weight vector in the rational representation V_λ of $GL(3)$ with highest weight λ. We describe V_λ and $\mathcal{O}_\lambda$ explicitly in Section 3. The action of $a \times g = (a, g) \in F^\times \times GL(3)$ on a function $\varphi \in \mathcal{S}(\mathcal{O}_\lambda)$ is

$$(a \times g)\varphi(x) = \varphi(a g^{-1} x), \qquad x \in V_\lambda; \tag{1.3}$$

note that $\mathcal{O}_\lambda \subset V_\lambda$ is stable under scaling by an element $a \in F^\times$. So we are led to look for candidates for V such that ${}^N V$ is isomorphic to $\mathcal{S}(\mathcal{O}_\lambda)$.

Taking a hint from the well-known theta-correspondence of $SL(2) \times O(2n)$, we try to realize V as $\mathcal{S}(X)$, where X has the following properties:

(1) X carries an action of $F^\times \times GL(3)$.
(2) There exists a $GL(3)$-invariant regular function f on X of "degree 2" with respect to $F^\times$, i.e.,

$$f(agx) = a^2 f(x), \qquad \text{for } a\in F^\times,\ g\in GL(3), \text{ and } x\in X. \tag{1.4}$$

(3) The zero set of f, $X_0 = \{x\in X \mid f(x)=0\}$, is isomorphic to $\mathcal{O}_\lambda$, as a variety with an action of $F^\times\times GL(3)$.

Let $e_F : F\to \mathbf{C}^\times$ be a nontrivial additive character of F. We define a representation of $B\times GL(3)$ on $\mathcal{S}(X)$ by

$$(\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix}\times g)\varphi(x) = e_F(abf(x))\varphi(ag^{-1}x). \tag{1.5}$$

In keeping with our wish to suppress normalization factors and other technical details in this motivational section, the above equation is not entirely accurate. But note that the reason for requiring f to have degree 2 is so that 1.5 gives a genuine representation of $B\times GL(3)$. Then, heuristically, the space of N-coinvariants ${}^N\mathcal{S}(X)$ should be isomorphic to $\mathcal{S}(X_0)$. Indeed, ${}^N\mathcal{S}(X)$ is dual to the space of distributions that are invariant under N; by 1.5, these distributions should have support on those x such that for all $b\in F$, $e_F(bf(x))=1$; in other words, on the zero set of f. So if we can extend the above action of $B\times GL(3)$ to all of $SL(2)\times GL(3)$, we obtain a representation that "generically" realizes the local adjoint lift. It should be noted that the above ideas are not limited to $GL(3)$ but can be generalized to other reductive groups G; one just needs to replace the rational character λ of the Borel subgroup of $GL(3)$ with a rational character of a parabolic subgroup $P\subset G$, such that P is the stabilizer of the line through a highest weight vector in V_λ. Assuming that one can find a suitable X and extend the resulting representation of $B\times G$ to one of $SL(2)\times G$, one then obtains a correspondence between PS_2^χ and the induced representation $\operatorname{Ind}_P^G \chi\circ\lambda$.

We now describe our actual results, which, as mentioned above, are only for the case $F=\mathbf{R}$. In that case, we can obtain results by using the Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{gl}(3)$, or more generally $\mathfrak{sl}(2)$ and $\mathfrak{g}$, and working with infinitesimal representations of these Lie algebras, instead of with representations of the actual groups. We begin in Section 2 with a method to extend a representation of the Lie algebra $\mathfrak{b}$ of B to a representation of $\mathfrak{sl}(2)$, if one knows the action of the Casimir element $C_{\mathfrak{sl}(2)}$ of the universal enveloping algebra $U(\mathfrak{sl}(2))$. If one can find a suitable X as suggested by 1–3 above, this method then allows us to construct a candidate representation of $\mathfrak{sl}(2)\times\mathfrak{g}$ on a certain space of functions on X. The idea is to start with the representation of $\mathfrak{b}\times\mathfrak{g}$ resulting from 1.5, suitably match up the action of the Casimir elements $C_{\mathfrak{sl}(2)}\in U(\mathfrak{sl}(2))$ and $C_{\mathfrak{g}}\in U(\mathfrak{g})$, and use the aforementioned method to obtain the action of all of $\mathfrak{sl}(2)$. This is done in Section 3, where as an interesting (and perhaps unexpected) technical simplification we do not concentrate on the principal series representations PS_2^χ of $\mathfrak{sl}(2)$, but instead concentrate on making the discrete series representations DS_{k+2} of $\mathfrak{sl}(2)$ correspond to *finite-dimensional* representations of $\mathfrak{g}$. This essentially amounts to working with the compact torus $\{\left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)\}\subset SL(2,\mathbf{R})$, instead of the split torus $\{\left(\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right)\}$. The role played by the Jacquet module in the preceding motivational discussion is now played by the set of vectors annihilated by a lowering operator ε for the action of the compact torus. We specialize to the case $\mathfrak{g}=\mathfrak{gl}(3)$ and then describe a suitable approximation X' to the space X that we really want, and exhibit the desired representation of $\mathfrak{sl}(2) \times \mathfrak{g}$. This representation yields a correspondence that is still a form of the adjoint lift. Namely, it pairs off discrete series representations of $\mathfrak{sl}(2)$ coming from holomorphic modular forms of weight $k+2$ to finite-dimensional representations of $\mathfrak{g}$ with highest weight $k\rho$, where ρ is half the sum of the positive roots. On the level of groups, this pairs off discrete series representations of $SL(2,\mathbf{R})$ with representations of the *compact* group $U(3)$. The resulting correspondence is given by the map of L-groups described by the following slight variation on the adjoint map from $PGL(2)$ to $GL(3)$:

$$(1.6) \qquad {}^L(SL(2)) = PGL(2,\mathbf{C}) \hookrightarrow SO(3,\mathbf{C}) \to GL(3,\mathbf{C}) \rtimes \mathbf{Z}/2\mathbf{Z} = {}^L(U(3)).$$

In Section 4 we finally produce the "correct" form of $X_{\mathbf{R}}$, which is no longer quite an algebraic variety, but which is still a manifold. We then realize our representation by explicit unitary transformations on $L^2(X_{\mathbf{R}})$. As in our motivational section, it is easy to write down the action of $B \times U(3)$, or more precisely of its identity component $B^+ \times U(3)$, on $L^2(X_{\mathbf{R}})$. The formulas are more or less the ones in 1.5, and we again extend to an infinitesimal representation of $\mathfrak{sl}(2) \times \mathfrak{g}$ using the results of Section 2. We spend most of Section 4 deriving an explicit integral kernel for the operator giving the action of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and B^+ generate $SL(2)$, we have thus obtained explicit formulas for our representation. The action of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is similar to that of a Fourier transform; we essentially assemble the integral kernel piece by piece, by first calculating the effect of this analog of the Fourier transform on those functions analogous to the product of a Gaussian $e^{-\pi \sum x_j^2}$ with a polynomial in the $\{x_j\}$. We conclude Section 4 with some discussion of possible directions for further investigation, which can be viewed as a continuation of this introduction. As our motivation for conducting this investigation comes from automorphic forms, we touch on what might be needed in order to be able to extend our construction p-adically, and (although this seems distinctly more difficult) to be able to prove that the resulting local constructions, if they exist, fit together in a globally coherent way.

It is a pleasure to acknowledge several helpful conversations and much advice from a number of people while this work was in progress. Often the influence was indirect: a chance comment from one of these people about a related topic suddenly clarified matters at a point when I was stuck on this calculation. Of course, any errors in this paper are my own. I found a preprint [**Bl**] by and discussions with A. Bluher helpful in understanding the Fock-Bargmann model. R. Brylinski explained to me some points about geometric quantization and gave me reprints of her work (e.g., [**BK**], and the references in it) with B. Kostant; although this paper only makes minimal reference to their work, the overall picture of their work was quite inspiring and set me off on the right track. B. Gross and I had several enlightening and encouraging discussions, and the idea of taking G to be compact grew out of them; he also suggested the candidate for the p-adic form of $X_{\mathbf{R}}$ mentioned at the end of Section 4. D. Kazhdan also gave encouraging advice, and explained a lot in his lectures on representation theory and algebraic geometry at Harvard in the spring of 1996. R. Langlands made helpful suggestions about extending a representation from B to $SL(2)$, as did L. Mantini, who also explained some aspects of representation theory of Lie groups to me. K. Oden and I had several helpful conversations on the topic of spherical harmonics. D. Prasad made the crucial suggestion to look at the discrete series representations of $SL(2)$ instead of the principal series. S. Rallis pointed out a mistake in my early investigations and clarified the motivating construction described in the introduction. S. Sahi explained several points about representations of Lie groups, and pointed me in the direction of [**BSS**], which describes a different representation of $GL(2) \times GL(3)$ on the space of rank 1 matrices, which it was helpful to study. G. Savin and I discussed the material in this paper on more than one occasion, including the matching up of the actions of the centers of the universal enveloping algebras $Z(\mathfrak{sl}(2))$ and $Z(\mathfrak{g})$; his preprint [**S**] ultimately inspired a lot of the philosophy behind this paper. J. Stalker explained to me a fair amount of complex analysis. D. Vogan gave me some advice on "exponentiating" a representation of a Lie algebra to a Lie group, and also made some remarks that helped me understand the decomposition 4.13.

Finally, it is a pleasure and an honor to dedicate this paper to my Ph.D. advisor, Goro Shimura, for leading me into the subject of automorphic forms. In particular, he introduced me to the theta-correspondence between the double cover of $SL(2)$ and $O(2,1)$, which, as is well-known, gives rise to Shimura's correspondence [**Sh**] between modular forms of half-integral weight and of even weight.

2. Representations of $\mathfrak{sl}(2)$.

In this section we describe a way to extend representations of "part" of the Lie algebra $\mathfrak{sl}(2,\mathbf{C})$ to actual representations. Most importantly, we prove Lemma 2.2, which is crucial to this paper; here the simplicity of the expression for the Casimir element of $\mathfrak{sl}(2)$ plays an important role. To begin, we fix two bases $\{e,f,h\}$ and $\{\delta,\varepsilon,K\}$ for $\mathfrak{sl}(2)$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1}$$

$$\delta = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

$\mathbf{R}h$ and $\mathbf{R}K$ are the split and nonsplit toral subalgebras of $\mathfrak{sl}(2,\mathbf{R})$; $\mathbf{C}h + \mathbf{C}e$ and $\mathbf{C}K + \mathbf{C}\delta$ are corresponding Borel subalgebras of $\mathfrak{sl}(2,\mathbf{C})$. The Casimir element in the universal enveloping algebra $U(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$ is

$$C_{\mathfrak{sl}(2)} = h^2 - 2h + 4ef = -K^2 - 2iK + \delta\varepsilon. \tag{2.3}$$

Definition 2.1. Let X be a manifold. We define a *formal differential operator* on X to be a $\mathcal{C}^\infty$ differential operator defined on $X - X_0$, where X_0 is a finite union of lower-dimensional submanifolds of X. (X will usually be an algebraic variety, and a formal differential operator is then one defined at the generic point of X.)

Write $\mathcal{FD}(X)$ for the ring of formal differential operators on X. A *formal representation* of a Lie algebra $\mathfrak{g}$ on X is a homomorphism of Lie algebras $r : \mathfrak{g} \to \mathcal{FD}(X)$, or equivalently a ring homomorphism $r : U(\mathfrak{g}) \to \mathcal{FD}(X)$. Typically, X will be an algebraic variety, in which case $\mathfrak{g}$ acts on the field of rational functions on X.

Lemma 2.2. *Let $\mathfrak{b} = \mathbf{C}h + \mathbf{C}e$ be a Borel subalgebra of $\mathfrak{sl}(2)$. Let r be a formal representation of $\mathfrak{b}$ on a manifold X, and let $A \in \mathcal{FD}(X)$ be a formal differential operator. Suppose that*

1. *$r(e)$ is an invertible formal differential operator (typically, multiplication by some nonzero function on X);*

(2) *A commutes with the image of* r.

Then r *can be uniquely extended to a formal representation of* $\mathfrak{sl}(2)$ *on* X *in such a way that* $r(C_{\mathfrak{sl}(2)}) = A$.

PROOF. Essentially, we must define $r(f)$. One just solves for $r(f)$ in the equation

$$r(h)^2 - 2r(h) + 4r(e)r(f) = A = r(C_{\mathfrak{sl}(2)}). \tag{2.4}$$

This involves inverting $r(e)$. One immediately checks that this works: e.g., to verify that $[r(h), r(f)] = -2r(f)$, one evaluates

$$\begin{aligned} 0 = [r(h), A] &= 4\,[r(h), r(e)]\,r(f) + 4r(e)\,[r(h), r(f)] \\ &= 8r(e)r(f) + 4r(e)\,[r(h), r(f)], \end{aligned} \tag{2.5}$$

because r is a formal representation of $\mathfrak{b}$. One can then cancel the invertible operator $r(e)$. □

REMARK 2.3. An analogous result holds if one uses K and δ instead of h and e above. Here one solves for $r(\varepsilon)$ in

$$-r(K)^2 - 2ir(K) + r(\delta)r(\varepsilon) = A = r(C_{\mathfrak{sl}(2)}). \tag{2.6}$$

EXAMPLE 2.4 (The discrete series of $\mathfrak{sl}(2)$). Given an integer $k \geq 1$, there exists a unique irreducible Harish-Chandra module DS_k with an action of $\mathfrak{sl}(2)$, with the property that DS_k contains a vector v such that $\varepsilon v = 0$ and $Kv = -ikv$. Namely,

$$DS_k = \bigoplus_{l \geq 0} \mathbf{C} \cdot \delta^l v, \tag{2.7}$$

$$K(\delta^l v) = -i(k + 2l)\delta^l v, \tag{2.8}$$

$$\varepsilon(\delta^l v) = -4l(l + k - 1)\delta^{l-1} v. \tag{2.9}$$

DS_k is the type at ∞ of an automorphic representation of $GL(2, \mathbf{Q})$ corresponding to a classical holomorphic modular form of weight k. (To be precise, DS_1 is really a limit of discrete series; this distinction plays no role in this paper, and will henceforth be ignored.) We shall use Lemma 2.2 to obtain a "natural" realization of DS_k.

Let X be the open interval $(0, +\infty)$, fix $\alpha > 0$ (usually $\alpha = \pi$), and consider the formal representation r_k of $\mathfrak{sl}(2)$ acting on functions $\varphi(u)$ for $u \in X$, characterized by:

$$r_k(e) = i\alpha u \qquad (\text{i.e., } r_k(e)\varphi(u) = i\alpha u\varphi(u)); \tag{2.10}$$

$$r_k(h) = 2u\frac{\partial}{\partial u} + k; \tag{2.11}$$

$$r_k(C_{\mathfrak{sl}(2)}) = k(k-2); \tag{2.12}$$

$$r_k(f) = \frac{i}{\alpha}\left(u\frac{\partial^2}{\partial u^2} + k\frac{\partial}{\partial u}\right). \tag{2.13}$$

(The last equation follows from Lemma 2.2.)

PROPOSITION 2.5. r_k *leaves stable the space of functions on* $(0, +\infty)$ *of the form* $\varphi(u) = p(u)e^{-\alpha u}$, *where* p *is a polynomial. The representation of* r_k *on this space is isomorphic to* DS_k. *In particular, the function* $e^{-\alpha u}$ *corresponds to the vector* v *above.*

Proof. This boils down to checking that:

(1) $r_k(\varepsilon)(e^{-\alpha u}) = 0$;
(2) $r_k(K)(e^{-\alpha u}) = -ike^{-\alpha u}$;
(3) $r_k(\delta)^l(e^{-\alpha u})$ has the form $p_l(u)e^{-\alpha u}$, where $p_l(u)$ is a polynomial with leading term $(-4\alpha u)^l$.

□

So far we have described DS_k only as a representation of the Lie algebra $\mathfrak{sl}(2)$. As is well known, DS_k comes from a unitary representation of $SL(2,\mathbf{R})$. We now describe the action of the $SL(2,\mathbf{R})$ on the above model; this will be useful in Section 4. To do this, we recall the familiar theta-correspondence between $SL(2,\mathbf{R})$ and $O(2)$.

Theorem 2.6. *Fix a parameter $\alpha > 0$. Then there exists a unique unitary representation R of $SL(2,\mathbf{R}) \times O(2)$ on $L^2(\mathbf{R}^2)$, given by the formulas below. To simplify notation, we view $O(2)$ and $SL(2,\mathbf{R})$ as subgroups of the product; e.g., $R(g)$ below really refers to $R(1 \times g)$. We also view $v \in \mathbf{R}^2$ as a row vector, and write $\langle v, w\rangle$ for the inner product of v and w.*

$$R(g)f(v) = f(vg), \qquad \text{for } g \in O(2); \tag{2.14}$$

$$R(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix})f(v) = ae^{i\alpha ab\langle v,v\rangle} f(av); \tag{2.15}$$

$$R(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})f(v) = \frac{-i\alpha}{\pi} \int_{\mathbf{R}^2} f(w)e^{-2i\alpha\langle v,w\rangle}\, dw. \tag{2.16}$$

Here $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ are elements of $SL(2,\mathbf{R})$. Under the above action, $L^2(\mathbf{R}^2)$ decomposes into the Hilbert direct sum $\widehat{\bigoplus}_{k\geq 0} DS_{k+1} \otimes V_k$, where V_0 is the trivial representation of $O(2)$, and V_k, for $k \geq 1$, is the irreducible two-dimensional representation of $O(2)$ whose character χ_k, evaluated at $g = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \in O(2)$, is $\chi_k(g) = e^{ik\theta} + e^{-ik\theta}$.

Proof. This arises from restricting the metaplectic representation of Segal-Shale-Weil to the dual pair $SL(2,\mathbf{R}) \times O(2) \subset Sp(4)$. See for example [**JL**], Proposition 1.3 and Lemma 5.12. Note that matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ generate $SL(2,\mathbf{R})$. □

It follows from the above theorem that for $k \geq 1$, DS_k is isomorphic to the representation obtained from the action of $SL(2)$ on the subspace

$$L^2(\mathbf{R}^2)_{k-1} = \{f \in L^2(\mathbf{R}^2) \mid f(v_1, v_2) = (v_1 + iv_2)^{k-1}\varphi(v_1^2 + v_2^2)\}. \tag{2.17}$$

Requiring f to be in L^2 is equivalent to requiring $\varphi(u) \in L^2((0, +\infty), u^{k-1}\, du)$. As a matter of fact, the representation r_k of Proposition 2.5 is exactly the infinitesimal version of the restriction of R to $L^2(\mathbf{R}^2)_{k-1}$. The theta-correspondence thus gives a way of constructing r_k without using Lemma 2.2. We nonetheless wished to illustrate the use of Lemma 2.2 on a familiar example, before embarking on the main construction of this paper.

Let us call R_k the resulting representation of $SL(2)$ on $L^2((0,+\infty), u^{k-1}\,du)$. Here are explicit formulas for R_k, that we shall use in Section 4.

$$R_k(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix})\varphi(u) = a^k \varphi(a^2 u) e^{i\alpha abu}; \tag{2.18}$$

$$R_k(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})\varphi(u) = (-i)^k \alpha \int_0^\infty (ul)^{-(k-1)/2} J_{k-1}(2\alpha\sqrt{lu})\varphi(l) l^{k-1}\,dl. \tag{2.19}$$

Here J_n is the usual Bessel function

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\theta} e^{-in\theta}\,d\theta = \frac{i^n}{2\pi}\int_0^{2\pi} e^{-iz\cos\theta} e^{in\theta}\,d\theta. \tag{2.20}$$

(See (4.5–5) of [**H**].) This follows from expressing the action of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, essentially a Fourier transform, in polar coordinates.

Remark 2.7. Here is a heuristic way to understand 2.19, which may be of interest. Make the *Ansatz*

$$R_k(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})\varphi(u) = \int_0^\infty K(l,u)\varphi(l) l^{k-1}\,dl, \tag{2.21}$$

for a kernel function K. Now

$$R_k(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix})R_k(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) = R_k(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})R_k(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}), \tag{2.22}$$

so $K(l, a^2u) = K(a^2l, u)$; thus K can depend only on the product lu: $K(l,u) = K(lu)$. Furthermore,

$$R_k(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})r_k(e)R_k(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})^{-1} = r_k(-f); \tag{2.23}$$

hence $i\alpha lK(lu) = \frac{-i}{\alpha}(u\frac{\partial^2}{\partial u^2} + k\frac{\partial}{\partial u})K(lu)$, i.e., $zK''(z) + kK'(z) + \alpha^2 K(z) = 0$. Now $z^{-(k-1)/2}J_{k-1}(2\alpha\sqrt{z})$ is one solution of this differential equation (see [**H**], (4.5–4)), and the "other" solution behaves near $z = 0$ like z^{1-k} (or even $\log z$, if $k = 1$); presumably the first solution is the only one small enough to be the kernel of a meaningful integral operator on $L^2((0,+\infty), u^{k-1}\,du)$.

3. Generalized Fock-Bargmann space

We recall that the Fock-Bargmann space for the dual pair $SL(2)\times O(2n)$ is the set of holomorphic functions on $\mathbf{C}^{2n}$ which are L^2 with respect to a certain measure ([**F1**], sections 1.6 and 4.2; we restrict to even-dimensional orthogonal groups in order to avoid the double cover of $SL(2)$). Let R be the representation of $SL(2)\times O(2n)$ on this space. We shall restrict ourselves to the subspace of $SO(2)\times O(2n)$-finite vectors, which is simply the space $\mathbf{C}[X] = \mathbf{C}[z_1,\dots,z_{2n}]$ of polynomials on $X = \mathbf{C}^{2n}$. Then the element $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)\times g \in SO(2)\times O(2n)$ acts on $f\in\mathbf{C}[X]$ by

$$R(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\times g)\varphi(v) = e^{-in\theta}\varphi(e^{-i\theta}vg), \tag{3.1}$$

where $v = (z_1,\dots,z_{2n})\in X$ is viewed as a row vector. The resulting infinitesimal representation r of $\mathfrak{so}(2)\times\mathfrak{o}(2n)$ extends to one of $\mathfrak{sl}(2)\times\mathfrak{o}(2n)$. We give the action of $\mathfrak{sl}(2)$, using the notation from 2.2:

$$r(K)\varphi = -i(n + \sum_{j=1}^{2n} z_j \frac{\partial}{\partial z_j})\varphi, \tag{3.2}$$

$$r(\delta)\varphi = (\sum_{j=1}^{2n} z_j^2)\varphi, \tag{3.3}$$

$$r(\varepsilon)\varphi = -(\sum_{j=1}^{2n} \frac{\partial^2}{\partial z_j^2})\varphi. \tag{3.4}$$

The following theorem is very well known.

THEOREM 3.1. *Under the action of* $\mathfrak{sl}(2) \times \mathfrak{o}(2n)$,

$$\mathbf{C}[X] \cong \bigoplus_{k\geq 0} DS_{k+n} \otimes \mathcal{H}_k, \tag{3.5}$$

where $\mathcal{H}_k$ *is the representation of* $O(2n)$ *on degree* k *harmonic polynomials in* $\{z_1, \dots z_{2n}\}$. *Specifically,*

$$DS_{k+n} \otimes \mathcal{H}_k = \bigoplus_{l\geq 0} (\sum_{j=1}^{2n} z_j^2)^l \mathcal{H}_k. \tag{3.6}$$

Furthermore, write $Z(\mathfrak{sl}(2))$ *and* $Z(\mathfrak{o}(2n))$ *for the centers of the universal enveloping algebras* $U(\mathfrak{sl}(2))$ *and* $U(\mathfrak{o}(2n))$. *Then the images of* $r(Z(\mathfrak{sl}(2)))$ *and* $r(Z(\mathfrak{o}(2n)))$ *in* $\operatorname{End} \mathbf{C}[X]$ *are the same.*

In this section we present a generalization of the Fock-Bargmann model of Theorem 3.1, in which $O(2n)$ is replaced by a general compact Lie group G. To this end we observe two features of the above representation r. The first is that $r(Z(\mathfrak{o}(2n)))$ is the set of polynomials in $r(C_{\mathfrak{o}(2n)})$, where $C_{\mathfrak{o}(2n)}$ is the Casimir element in $U(\mathfrak{o}(2n))$. We shall expand on this point in Lemma 3.2. The second feature to observe is that the isomorphism 3.5 boils down to asserting that every element of $\mathbf{C}[X]$ is annihilated by a sufficiently high power of $r(e)$, and that $\mathbf{C}[X]/r(\delta)\mathbf{C}[X] \cong \bigoplus_{k\geq 0} W_{k+n} \otimes \mathcal{H}_k$ as an $SO(2) \times O(n)$ representation. Here $W_{k+n} = DS_{k+n}/\delta(DS_{k+n})$ is the one-dimensional $SO(2)$-representation on which $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ acts as multiplication by $e^{-i(k+n)\theta}$. The point to note here is that $\mathbf{C}[X]/r(\delta)\mathbf{C}[X] = \mathbf{C}[X]/(\sum_{j=1}^{2n} z_j^2)\mathbf{C}[X]$ is the same as the coordinate ring $\mathbf{C}[X_0]$ of $X_0 = \{(z_1, \dots, z_{2n}) \in X \mid \sum_{j=1}^{2n} z_j^2 = 0\}$. Moreover, X_0 is the Zariski closure of the orbit of any nonzero isotropic vector in $\mathbf{C}^{2n}$ under the complex group $O(2n, \mathbf{C})$. We shall come back to this point in Lemmas 3.4 and 3.5. For now, let us deal with the first observation above. Recall that $C_{\mathfrak{sl}(2)}$ is the Casimir element in $Z(\mathfrak{sl}(2))$; more generally, if $\mathfrak{g}$ is a complex reductive Lie algebra, write $C_{\mathfrak{g}}$ for the Casimir element in $Z(\mathfrak{g})$.

LEMMA 3.2. *Let* $\mathfrak{g}$ *be a complex reductive Lie algebra, with maximal torus* $\mathfrak{t}$. *Fix a set of positive roots and let* ρ *be half of their sum. Let* $0 \neq \lambda \in \mathfrak{t}^*$ *be a dominant weight with respect to the choice of positive roots, and assume that there exists an element* w *of the Weyl group* W *of* $\mathfrak{g}$, *such that*

(1) $w^2 = 1$;
(2) $w\lambda = -\lambda$;

(3) *one of the following conditions holds (they are all equivalent, given conditions 1 and 2):*
   (a) $\rho - w\rho$ *is a multiple of* λ,
   (b) w *stabilizes the line* $\{\rho + k\lambda \mid k \in \mathbf{R}\} \subset \mathfrak{t}^*$,
   (c) $w(\rho) = \rho - 2\frac{\langle \rho,\lambda\rangle}{\langle \lambda,\lambda\rangle}\lambda$ *(since w preserves the Killing form $\langle\ ,\ \rangle$).*

Assume that r is a representation of $\mathfrak{g}$ on a vector space V that decomposes into a direct sum of finite-dimensional irreducible representations, each of which has highest weight a multiple of λ. Then $Z(\mathfrak{g})$ acts on V by polynomials in $C_{\mathfrak{g}}$, i.e., the image $r(Z(\mathfrak{g})) \subset \operatorname{End} V$ consists of the polynomials in $r(C_{\mathfrak{g}})$.

Proof. Given $A \in Z(\mathfrak{g})$, we exhibit a polynomial $q(z)$ depending only on A and λ, but not on V, such that A and $q(C_{\mathfrak{g}})$ have the same action on V. This is an easy consequence of Harish-Chandra's theorem that $Z(\mathfrak{g})$ is isomorphic to the ring of W-invariant polynomial functions on $\mathfrak{t}^*$ (see, e.g., [**Hu**], section 23.3). Namely, A corresponds to the W-invariant polynomial function a on $\mathfrak{t}^*$ such that

$$v \in V_\mu \Rightarrow r_\mu(A)v = a(\mu + \rho)v, \tag{3.7}$$

where r_μ is a representation of $\mathfrak{g}$ with highest weight μ on the vector space V_μ (for instance, the Verma module or its irreducible quotient). Hence for all k, A acts on $V_{k\lambda}$ by the scalar $a(k\lambda + \rho)$, a polynomial function of k. Now a is W-invariant, so $a(k\lambda+\rho) = a(w(k\lambda+\rho))$, for the element $w \in W$ described above. Thus $a(k\lambda+\rho)$ is invariant under replacing k by $-k - 2\langle\rho,\lambda\rangle/\langle\lambda,\lambda\rangle$, and we obtain that

$$a(k\lambda + \rho) = q(\langle\lambda,\lambda\rangle k^2 + 2\langle\rho,\lambda\rangle k), \tag{3.8}$$

for some polynomial q. (This is analogous to saying that polynomials in z invariant under replacing z by $-z$ are themselves polynomial functions of z^2.) But $\langle\lambda,\lambda\rangle k^2 + 2\langle\rho,\lambda\rangle k$ is precisely the constant by which $C_{\mathfrak{g}}$ acts on $V_{k\lambda}$. Hence $A - q(C_{\mathfrak{g}})$ annihilates all the $V_{k\lambda}$, and hence annihilates V, as desired. □

Remark 3.3. We shall apply Lemma 3.2 to the case $\lambda = \rho$. Then one can take w to be the longest element $w_0 \in W$. (w_0 sends the set of positive roots to the set of negative roots, so $w_0\rho = -\rho$.) Another fundamental example is $\mathfrak{g} = \mathfrak{o}(n)$ and λ such that V_λ is the standard (n-dimensional) representation of $\mathfrak{o}(n)$. We leave it to the reader to find the corresponding w.

Lemma 3.4. *Let G be a complex reductive algebraic group, $\mathfrak{g}$ its Lie algebra, and $\lambda \in \mathfrak{t}^*$ a dominant integral weight. Let V_λ be the irreducible representation of G with highest weight λ. Let $\mathcal{O}_\lambda \subset V_\lambda$ be the orbit under G of a highest weight vector of V_λ, and call its Zariski closure $\overline{\mathcal{O}_\lambda}$. Then G acts on the coordinate ring $\mathbf{C}[\overline{\mathcal{O}_\lambda}]$ via $g\varphi(x) = \varphi(g^{-1}x)$. As a G-representation,*

$$\mathbf{C}[\overline{\mathcal{O}_\lambda}] \cong \bigoplus_{k\geq 0} V_{k\lambda}^*, \tag{3.9}$$

where $V_{k\lambda}^*$ is the representation of G dual to $V_{k\lambda}$. More precisely, $\overline{\mathcal{O}_\lambda}$ is a cone, and hence $\mathbf{C}[\overline{\mathcal{O}_\lambda}]$ inherits the structure of a graded algebra from the affine algebra $\mathbf{C}[V_\lambda]$ (which surjects onto $\mathbf{C}[\overline{\mathcal{O}_\lambda}]$). $V_{k\lambda}^*$ is then precisely the degree k part of $\mathbf{C}[\overline{\mathcal{O}_\lambda}]$.

Proof. See Proposition 2.5 of [**BK**] and the references cited there. Another way to view 3.9 is to consider the action of $\mathbf{G}_m \times G$ on $\overline{\mathcal{O}_\lambda}$. ($\mathbf{G}_m$ is the multiplicative group $\mathbf{C}^\times$, which acts on V_λ by scaling the vectors; this action preserves the cone $\overline{\mathcal{O}_\lambda}$.) Then under the resulting action of $\mathbf{G}_m \times G$, $\mathbf{C}[\overline{\mathcal{O}_\lambda}]$ decomposes into a direct sum of $W_k \otimes V^*_{k\lambda}$, where W_k is the one-dimensional representation of $\mathbf{G}_m$ on which $\zeta \in \mathbf{G}_m$ acts by multiplication by ζ^k. Also note that the grading on $\mathbf{C}[\overline{\mathcal{O}_\lambda}]$ arises from the action of $\mathbf{G}_m$ on $\overline{\mathcal{O}_\lambda}$. $\square$

Lemma 3.5. *In the situation of Lemma 3.4, suppose given an irreducible affine algebraic variety X on which $\mathbf{G}_m \times G$ acts, such that X is a cone on which $\mathbf{G}_m$ acts by dilations (so given $\zeta \in \mathbf{G}_m$ and $x \in X$, one obtains $\zeta x \in X$). Further assume that*

(1) *The action of $\mathbf{G}_m$ turns the coordinate ring $\mathbf{C}[X]$ into a graded algebra in degrees ≥ 0: i.e., $\mathbf{C}[X] = \bigoplus_{k \geq 0} \mathbf{C}[X]_k$, where*

$$\text{for } \varphi_k \in \mathbf{C}[X]_k,\ x \in X,\ \text{and } \zeta \in \mathbf{G}_m, \qquad \varphi_k(\zeta x) = \zeta^k \varphi(x). \tag{3.10}$$

(2) *There exists $d > 0$ and a degree d G-invariant function $f \in \mathbf{C}[X]_d$ such that the zero set $X_0 = \{x \in X \mid f(x) = 0\}$ of f is isomorphic to $\overline{\mathcal{O}_\lambda}$, as a variety with an action of $\mathbf{G}_m \times G$.*

Then, as a representation of G,

$$\mathbf{C}[X]_k \cong \bigoplus_{0 \leq l \leq k/d} V^*_{(k-ld)\lambda}. \tag{3.11}$$

Equivalently, let $W_k \subset \mathbf{C}[X]_k$ be the unique subspace isomorphic to $V^*_{k\lambda}$. Then

$$\mathbf{C}[X] = \bigoplus_{k,l \geq 0} f^l W_k, \qquad f^l W_k \cong V^*_{k\lambda}, \qquad f^l W_k \subset \mathbf{C}[X]_{k+ld}. \tag{3.12}$$

Proof. Observe that $\mathbf{C}[X_0]$ is $\mathbf{C}[X]/f\mathbf{C}[X]$. By Lemma 3.4, its degree k part, $\mathbf{C}[X_0]_k$, is isomorphic to $V^*_{k\lambda}$. On the other hand, we have the short exact sequence of G-modules

$$0 \longrightarrow \mathbf{C}[X]_{k-d} \xrightarrow{\cdot f} \mathbf{C}[X]_k \longrightarrow \mathbf{C}[X_0]_k \longrightarrow 0, \qquad \mathbf{C}[X_0]_k \cong V^*_{k\lambda} \tag{3.13}$$

where $\mathbf{C}[X]_{k-d}$ is taken to be 0 if $k - d < 0$. The above sequence splits, as G is reductive, and we obtain 3.11 by induction. (We used the assumption that X was irreducible to conclude that multiplication by f was injective.) $\square$

Corollary 3.6. *In the situation of Lemma 3.5, assume that λ satisfies the conditions of Lemma 3.2. Consider the representation of $\mathfrak{g}$ on $\mathbf{C}[X]$. Then $Z(\mathfrak{g})$ acts on $\mathbf{C}[X]$ by polynomials in $C_\mathfrak{g}$.*

Proof. This is essentially immediate from Lemmas 3.2 and 3.5, but one must pay attention to the distinction between $V_{k\lambda}$ and $V^*_{k\lambda}$. Namely, let w_0 be the longest element in W. Then $V^*_{k\lambda} \cong V_{k\lambda'}$, where $\lambda' = w_0(-\lambda)$. λ' then also satisfies the conditions of Lemma 3.2, using $w_0 w w_0^{-1}$. Incidentally, note that $\langle \lambda', \lambda' \rangle = \langle \lambda, \lambda \rangle$, and hence $C_\mathfrak{g}$ acts by the same scalar on $V_{k\lambda}$ and $V^*_{k\lambda}$. $\square$

Theorem 3.7 (Generalized Fock-Bargmann model). *Let X be a variety as in Lemma 3.5, and λ be as in Lemma 3.2. Assume furthermore that the G-invariant function $f \in \mathbf{C}[X]$ has degree $d = 2$, and that $k_0 = 1 + \frac{\langle \rho, \lambda \rangle}{\langle \lambda, \lambda \rangle}$ is an integer. Then there exists a representation r of $\mathfrak{sl}(2) \times \mathfrak{g}$ on $\mathbf{C}[X]$, such that*

(1) *$r(\delta)\varphi = f\varphi$, for $\varphi \in \mathbf{C}[X]$.*
(2) *$r(K)\varphi = -i(k_0 + E)\varphi$, where $E\varphi(x) = \frac{\partial}{\partial \zeta}\Big|_{\zeta=1} \varphi(\zeta x)$. (E is usually called the Euler operator on X.)*

(3) $\mathfrak{g}$ *acts by the infinitesimal action arising from the action of* G.
(4) $\mathbf{C}[X] \cong \bigoplus_{k\geq 0} DS_{k+k_0} \otimes V^*_{k\lambda}$.

Moreover, the images $r(Z(\mathfrak{sl}(2)))$ *and* $r(Z(\mathfrak{g}))$ *of the centers of the universal enveloping algebras in* End $\mathbf{C}[X]$ *coincide. In other words,* $Z(\mathfrak{g})$ *acts by polynomials in* $C_{\mathfrak{sl}(2)} \in Z(\mathfrak{sl}(2))$.

PROOF. $C_{\mathfrak{g}}$ acts on $V^*_{k\lambda}$ by the scalar $\langle\lambda,\lambda\rangle k^2+2\langle\rho,\lambda\rangle k = \langle\lambda,\lambda\rangle(k+k_0)(k+k_0-2)-\langle\lambda,\lambda\rangle k_0(k_0-2)$. On the other hand, $C_{\mathfrak{sl}(2)}$ acts on DS_{k+k_0} by $(k+k_0)(k+k_0-2)$. Thus the desired representation r must satisfy

(5) $r(C_{\mathfrak{sl}(2)}) = \frac{1}{\langle\lambda,\lambda\rangle} r(C_{\mathfrak{g}}) + k_0(k_0-2)$.

We now build up the representation r in stages. First note that requirements 1 and 2 define a representation of the Borel subalgebra $\mathfrak{b} = \mathbf{C}K + \mathbf{C}\delta$ of $\mathfrak{sl}(2)$ on $\mathbf{C}[X]$; this is where we need f to have degree 2. Next define the action of $\mathfrak{g}$ by requirement 3, and note that the actions of $\mathfrak{b}$ and of $\mathfrak{g}$ commute (since the actions of $\mathfrak{b}$ and G commute). The action of $\mathfrak{g}$ tells us how the operator $r(C_{\mathfrak{g}})$ acts, and hence by requirement 5 we obtain the action of $C_{\mathfrak{sl}(2)}$. Note that $r(C_{\mathfrak{sl}(2)})$ therefore commutes with the action of $\mathfrak{b}$, since $r(C_{\mathfrak{sl}(2)})$ can be expressed in terms of $\mathfrak{g}$. Of course, $r(C_{\mathfrak{sl}(2)})$ also commutes with the action of $\mathfrak{g}$, because $r(C_{\mathfrak{g}})$ does.

So far we have obtained a representation of $\mathfrak{b}\times\mathfrak{g}$ on $\mathbf{C}[X]$, and have an operator $r(C_{\mathfrak{sl}(2)})$ that commutes with the action of $\mathfrak{b}$ and $\mathfrak{g}$. We now use Lemma 2.2 to extend the action of $\mathfrak{b}$ to a formal representation of all of $\mathfrak{sl}(2)$. By the proof of that lemma, the resulting formal action of $\mathfrak{sl}(2)$ will commute with the existing action of $\mathfrak{g}$. By Corollary 3.6, $Z(\mathfrak{g})$ acts on $\mathbf{C}[X]$ by polynomials in $C_{\mathfrak{g}}$, and this action can be reexpressed in terms of $C_{\mathfrak{sl}(2)}$, using requirement 5. We now show that requirement 4 holds, and with it, the fact that r is a genuine (and not just formal) representation of $\mathfrak{sl}(2)$; specifically, we must check that $r(\varepsilon)$ sends $\mathbf{C}[X]$ to itself, instead of introducing a denominator of f. Let $W_k \subset \mathbf{C}[X]_k$ be the subspace isomorphic to $V^*_{k\lambda}$, as guaranteed by Lemma 3.5. Then $\bigoplus_{l\geq 0} f^l W_k$ is the $V^*_{k\lambda}$-isotypic component of $\mathbf{C}[X]$ under the action of G. Let us check that $r(\varepsilon)$ preserves this isotypic component. We have concocted r so that $C_{\mathfrak{sl}(2)}$ acts on this isotypic component by the scalar $(k+k_0)(k+k_0-2)$. We also see explicitly that if $\varphi \in W_k$, then the action of K on a typical element $f^l\varphi$ in this isotypic component is

$$r(K)(f^l\varphi) = -i(k_0+k+2l)f^l\varphi. \tag{3.14}$$

Lastly, $r(\delta)$ is multiplication by f. Putting this together and using 2.6 from the proof of Lemma 2.2, we obtain that

$$r(\varepsilon)(f^l\varphi) = -4l(l+k+k_0-1)f^{l-1}\varphi \begin{cases} \in f^{l-1}W_k & \text{if } l \geq 1, \\ = 0 & \text{if } l = 0. \end{cases} \tag{3.15}$$

It follows immediately that the isotypic component, $\bigoplus_{l\geq 0} f^l W_k$, is stable under $\mathfrak{sl}(2)\times\mathfrak{g}$. By comparison with 2.7–2.9, this isotypic component is isomorphic to $DS_{k+k_0}\otimes W_k$, as desired. □

REMARK 3.8. The hypothesis that k_0 be an integer merely serves to ensure that the discrete series representations DS_{k+k_0} come from genuine representations of $SL(2)$ and not, say, of its double cover.

REMARK 3.9. One might say that the above representation was one of $\mathfrak{sl}(2)\times G$, since the action of $\mathfrak{g}$ comes from one of the group G. The notation $\mathfrak{sl}(2)\times G$ is perhaps illogical, since it refers neither to a Lie algebra nor to a Lie group. Also note that $\mathbf{C}[X]$ is an explicit realization of the Harish-Chandra module of the unitary representation of $SL(2,\mathbf{R}) \times G_c$ on $\widehat{\bigoplus}_{k\geq 0} DS_{k+k_0} \otimes V^*_{k\lambda}$, when G_c is the compact real form of G.

We give three examples of the results of this section.

EXAMPLE 3.10. This is essentially Theorem 3.1. Let $G = O(2n)$, let V_λ be its standard $2n$-dimensional representation, and let $X = V_\lambda$ with the obvious action of $\mathbf{G}_m \times G$. Not surprisingly, we take $f \in \mathbf{C}[X]_2 = \mathbf{C}[V_\lambda]_2$ to be the invariant quadratic form. Then the zero set of f is the set of isotropic vectors, and this is precisely $\overline{\mathcal{O}_\lambda}$. Then, as we have seen, $\mathbf{C}[X]$ decomposes under G into $\bigoplus_{k,l\geq 0} f^l \mathcal{H}_k$, where $\mathcal{H}_k \cong V^*_{k\lambda} \cong V_{k\lambda}$. The action of $\mathbf{G}_m \times G$ on $\mathbf{C}[X]$ extends to the aforementioned action of $\mathfrak{sl}(2) \times G$ and yields 3.5. We have therefore reconstructed a proof of Theorem 3.1.

EXAMPLE 3.11. This example works for arbitrary G, and seems rather naive: one just takes the space X to be the product of $\overline{\mathcal{O}_\lambda}$ with an affine line, so $\mathbf{C}[X] = \mathbf{C}[\overline{\mathcal{O}_\lambda}][t]$. Here G acts on the first factor alone, and $\mathbf{G}_m$ acts on both factors by acting on $\overline{\mathcal{O}_\lambda}$ in the usual way, and on the affine line by dilations (i.e., in the resulting grading on $\mathbf{C}[X]$, t has degree 1.) Then one can take the G-invariant function f to be t. Then so long as λ is suitable (e.g., if $\lambda = \rho$), as mentioned in Remark 3.3), then one can apply Lemmas 3.4 and 3.5. One can modify this slightly to have $\zeta \in \mathbf{G}_m$ act on the affine line by dilating it by ζ^2; then t has degree 2 and one can conclude the result of Theorem 3.7. In the case of $G = O(2n)$ and V_λ the standard representation, it would be interesting to compare the explicit form of the representation of $\mathfrak{sl}(2) \times G$ from Theorem 3.1 with the one obtained in this setting.

EXAMPLE 3.12. This example concerns us most in this paper. Let $G = GL(3)$, and let $\lambda = \rho$. Then $V_\lambda = V_\rho$ happens to be isomorphic to the 8-dimensional adjoint representation of G on $\mathfrak{sl}(3)$, and is self-dual via the Killing form. Explicitly, $\mathcal{O}_\lambda$ is the set of 3×3 complex matrices of trace 0 and rank 1, with G acting by conjugating the matrices, and $\mathbf{G}_m$ acting by scaling them. $\overline{\mathcal{O}_\lambda}$ is just $\mathcal{O}_\lambda \cup \{0\}$, and so consists of matrices of trace 0 and rank at most 1. Instead of picking X as in Theorem 3.7, we take a substitute X' to be the space of 3×3 complex matrices of rank at most 1, but with arbitrary trace. Then $\mathbf{C}[X'] = \mathbf{C}[\{x_{ij} \mid 1 \leq i,j \leq 3\}]/I$, where I is the ideal generated by the 2×2 minors of the matrix $(x_{ij})_{1\leq i,j\leq 3}$. With this choice of X' we can take $t = x_{11} + x_{22} + x_{33}$, the trace function, to be our G-invariant function; it then has degree 1. We then obtain the following theorem.

THEOREM 3.13. *Let $G = GL(3)$, let X' be the above space, and let $t \in \mathbf{C}[X']$ be the trace function. Then $Z(\mathfrak{g})$ acts on $\mathbf{C}[X']$ by polynomials in $C_\mathfrak{g}$. Furthermore, $\mathbf{C}[X'] = \bigoplus_{k,l\geq 0} t^l W_k$, where $W_k \subset \mathbf{C}[X']_k$ is the subspace isomorphic to $V_{k\rho}$.*

REMARK 3.14. There is no need to distinguish between $V_{k\rho}$ and its dual, as they are isomorphic by the observation in the proof of Corollary 3.6. Note that $\dim V_{k\rho} = (k+1)^3$ by the Weyl character formula, and that these representations are all orthogonal. For other G (e.g., $SL(6)$), V_ρ may well be a symplectic representation of G.

REMARK 3.15. $Z(\mathfrak{g}) = Z(\mathfrak{gl}(3))$ is generated by elements in degrees 1, 2, and 3. The degree 1 element is the infinitesimal action of the center of G; it acts by 0 on

$\mathbf{C}[X']$, since $V_{k\rho}$ is actually a representation of $PGL(3)$. The degree 2 element is $C_{\mathfrak{g}}$. The degree 3 element can be written as the symmetrization of the determinant polynomial in the basis elements of $\mathfrak{gl}(3)$; it acts on $V_{k\rho}$ by 0.

The result of Theorem 3.13 does not immediately lead to a representation of $\mathfrak{sl}(2)\times\mathfrak{g}$. Indeed, our generalization of Fock-Bargmann space requires a G-invariant function of degree 2; so we try $f = t^2$, i.e., $f(x) = \operatorname{tr} x^2 = (\operatorname{tr} x)^2$ (this last equality follows from the fact that x has rank at most 1). Unfortunately, the zero set of $f = t^2$ is then no longer reduced. We nonetheless proceed to salvage something.

THEOREM 3.16. *Let $G = GL(3)$ and let X', t, and $f = t^2$ be as above. Let $\lambda = \rho$ and $k_0 = 2$. Then there exists a formal representation r of $\mathfrak{sl}(2)\times\mathfrak{g}$ (or even of $\mathfrak{sl}(2)\times G$, in the notation of Remark 3.9) on $\mathbf{C}[X']$, satisfying requirements 1–3 and 5 of Theorem 3.7, such that furthermore*

1. *$Z(\mathfrak{sl}(2))$ and $Z(\mathfrak{g})$ have the same image in the ring of formal differential operators $\mathcal{FD}(X')$.*
2. *$\mathfrak{sl}(2)\times\mathfrak{g}$ genuinely operates on the ring $\mathbf{C}[X'][t^{-1}]$ (the coordinate ring on $X' - \overline{\mathcal{O}_\lambda}$).*
3. *Let W_k be as in Lemma 3.5. Define the subspace (not subring) $V \subset \mathbf{C}[X']$ by*

$$V = \bigoplus_{k,l\geq 0} f^l W_k = \bigoplus_{k,l\geq 0} t^{2l} W_k. \tag{3.16}$$

Then V is stable under $\mathfrak{sl}(2)\times\mathfrak{g}$ and decomposes into $V \cong \bigoplus_{k\geq 0} DS_{k+2}\otimes V_{k\rho}$. Furthermore, we can view V as the Harish-Chandra module of a representation of $SL(2)\times U(3)$, as in Remark 3.9.

PROOF. This is analogous to the proof of Theorem 3.7. We simply find $r(\varepsilon)$ by matching up Casimir operators as usual; this involves dividing by f at some point. □

REMARK 3.17. The reader may be interested in an explicit description of r. To this end, let us introduce coordinates on X'. There is a surjective map $c : \mathbf{C}^3\times\mathbf{C}^3 \to X'$, namely $c(\vec{x},\vec{\xi}) = \vec{x}\,{}^t\vec{\xi}$:

$$c(x_1,x_2,x_3,\xi_1,\xi_2,\xi_3) = \begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}\begin{pmatrix}\xi_1 & \xi_2 & \xi_3\end{pmatrix}. \tag{3.17}$$

This allows us to view $\varphi \in \mathbf{C}[X']$ as a function on $\mathbf{C}^3\times\mathbf{C}^3$ such that for all $\lambda\in\mathbf{C}^\times$, $\varphi(\vec{x},\lambda\vec{\xi}) = \varphi(\lambda\vec{x},\vec{\xi})$. We shall describe operators on X' by giving operators in the variables x_i and ξ_i that preserve functions of the above type. We then have that

$$r(A)\varphi(\vec{x},\vec{\xi}) = \frac{\partial}{\partial\zeta}\Big|_{\zeta=0}\varphi(\exp(-\zeta A)\vec{x},\exp(\zeta\,{}^tA)\vec{\xi}), \qquad (A\in\mathfrak{g}); \tag{3.18}$$

$$r(K)\varphi = -i(2+\sum_{j=1}^{3} x_j\frac{\partial}{\partial x_j})\varphi = -i(2+\sum_{j=1}^{3}\xi_j\frac{\partial}{\partial \xi_j})\varphi; \tag{3.19}$$

$$r(\delta)\varphi = (x_1\xi_1+x_2\xi_2+x_3\xi_3)^2\varphi = t^2\varphi; \tag{3.20}$$

$$r(\varepsilon)\varphi = -(x_1\xi_1+x_2\xi_2+x_3\xi_3)^{-1}\left(\frac{\partial^2}{\partial x_1\partial\xi_1}+\frac{\partial^2}{\partial x_2\partial\xi_2}+\frac{\partial^2}{\partial x_3\partial\xi_3}\right)\varphi. \tag{3.21}$$

Note that the expression for $r(\varepsilon)$ involves a denominator of only t, not of t^2, even though we had to invert the operator $r(\delta)$ to obtain $r(\varepsilon)$ in the proof of Lemma 2.2. Equation 3.18 is the result of differentiating the action of G on functions on X', given by $R(g)\varphi(\vec{x},\vec{\xi}) = \varphi(g^{-1}\vec{x}, {}^t g\vec{\xi})$; viewing elements of X' as matrices, this amounts to the same as 4.5 below.

As noted above, one must select the subspace V, which is "half" of $\mathbf{C}[X']$, to serve as the space on which r is a genuine representation. Heuristically one may want to find a space X which is "half" of X', such that V is somehow the space of functions on X. This will come out more cleanly in the following section.

4. The Schrodinger model for $SL(2) \times U(3)$

We recall that the Fock-Bargmann space for $SL(2) \times O(2n)$ does not conveniently generalize to the p-adics. However, the Schrodinger model of that representation does so quite satisfactorily, as originally shown by Weil [**W**]. Over $\mathbf{R}$, this model is explicitly realized as an action of $SL(2,\mathbf{R}) \times O(2n)$ on $L^2(\mathbf{R}^{2n})$. For general n, the formulas are essentially identical to those given in Theorem 2.6, and make sense p-adically to give a representation of $SL(2,\mathbf{Q}_p) \times O(2n,\mathbf{Q}_p)$ on $L^2(\mathbf{Q}_p^{2n})$. (The formulas involve the function $e(x) = e^{2i\alpha x}$, usually with $\alpha = \pi$; this gets replaced by an additive character $e_p : \mathbf{Q}_p \to \mathbf{C}^\times$.) Our aim in this section is to construct an analog of the Schrodinger model for the representation of $SL(2,\mathbf{R}) \times U(3)$ described in Theorem 3.16, such that this Schrodinger-like model has some chance of eventually being generalized to the p-adics. At the moment, though, we can only produce the representation over $\mathbf{R}$ and speculate on possible further progress. The shift from $G = GL(3)$ to the *compact* form $G_{\mathbf{R}} = U(3)$ allows us to write down many explicit formulas that are analogous to classical calculations with spherical harmonics and representations of $O(2n)$. We note that our representation is unitary, as in the case of the classical Schrodinger model. Our results are contained in Theorems 4.2 and 4.9.

Our starting point is an infinitesimal realization of a Schrodinger model as a unitary representation of $\mathfrak{sl}(2) \times \mathfrak{u}(3)$ on the space of smooth vectors in a certain L^2 space. We shall then explain how to (indirectly) exponentiate this representation up to the groups $SL(2) \times U(3)$. This essentially boils down to computing one operator analogous to the Fourier transform in the case of $SL(2) \times O(2n)$. We first describe the space of functions on which our representation acts. By analogy with the Fock-Bargmann model, we try a space of functions on a variety $X'_{\mathbf{R}}$ defined over $\mathbf{R}$, with an action of $G_{\mathbf{R}} = U(3)$, such that the complexification X' of $X'_{\mathbf{R}}$ is the space in Example 3.12. Rather, we only want "half" of $X'_{\mathbf{R}}$.

Definition 4.1. From now on we shall drop the subscript $\mathbf{R}$ from the group G, which now denotes the group $U(3) = \{g \in GL(3,\mathbf{C}) \mid {}^t\overline{g} = g^{-1}\}$. We shall also define $X'_{\mathbf{R}}$ to be the space of Hermitian matrices of rank at most 1:

$$X'_{\mathbf{R}} = \{3 \times 3 \text{ complex matrices } M \text{ with } \overline{{}^tM} = M \text{ and } \operatorname{rank} M \leq 1\}. \tag{4.1}$$

(We could have scaled by i and had $X'_{\mathbf{R}}$ be the rank one antihermitian matrices instead, if we had wished to emphasize the parallelism between the set of rank 1 elements of $\mathfrak{gl}(3)$ from the previous section, and the set of rank 1 elements of $\mathfrak{u}(3)$ in this section. Our present choice of $X'_{\mathbf{R}}$ simplifies the notation somewhat.) We also define $X_{\mathbf{R}}$ to be the following half of $X'_{\mathbf{R}}$ (note that the trace of a Hermitian matrix is a real number):

$$
\begin{aligned}
X_{\mathbf{R}} &= \{M \in X'_{\mathbf{R}} \mid \operatorname{tr} M \geq 0\} \\
&= \{M \text{ of the form } \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \overline{z}_3 \end{pmatrix} \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix}\} \\
&\cong \mathbf{C}^3/\mathbf{T}.
\end{aligned} \tag{4.2}
$$

Here $\mathbf{T} = \{\zeta \in \mathbf{C}^\star \mid |\zeta| = 1\}$ is the group of complex numbers of norm one.

We define the action of G on $X'_{\mathbf{R}}$ and $X_{\mathbf{R}}$ by conjugation as before; again, it commutes with the action of $\mathbf{G}_m = \mathbf{R}^\times$ by scaling. We again take the G-invariant quadratic function $f(M) = \operatorname{tr} M^2 = (\operatorname{tr} M)^2$. Note that f is now positive definite (being $(|z_1|^2 + |z_2|^2 + |z_3|^2)^2$) and "anisotropic," in analogy to the case of the compact group $O(2n)$ and its invariant quadratic form $\sum x_j^2$ on $\mathbf{R}^{2n}$. We define a G-invariant measure dM on $X_{\mathbf{R}}$ by writing $z_j = x_j + y_j$ and defining

$$
\begin{aligned}
&\int_{X_{\mathbf{R}}} F(M)\, dM \\
&= \int_{\mathbf{C}^3} F(\begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \overline{z}_3 \end{pmatrix} \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix})(|z_1|^2 + |z_2|^2 + |z_3|^2)\, dx_1 dy_1 dx_2 dy_2 dx_3 dy_3.
\end{aligned} \tag{4.3}
$$

We then obtain a unitary representation R of $B^+ \times G$ on $L^2(X_{\mathbf{R}}, dM)$, where $B^+ = \{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a > 0\}$ is the connected component of the identity in the Borel subgroup of $SL(2, \mathbf{R})$:

$$
R(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix})F(M) = e^{\pi i a b \operatorname{tr} M^2} a^2 F(aM), \qquad \text{for } \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in B^+; \tag{4.4}
$$

$$
R(g)F(M) = F(g^{-1} M g), \qquad \text{for } g \in G. \tag{4.5}
$$

We aim to extend R to all of $SL(2, \mathbf{R}) \times G$. To this end, we begin by finding an "infinitesimal extension" to a representation r of the Lie algebras $\mathfrak{sl}(2) \times \mathfrak{g}$.

As a first step, R gives rise to a representation of the Lie algebras $\mathfrak{b} \times \mathfrak{g}$, which we extend to a formal representation r of $\mathfrak{sl}(2) \times \mathfrak{g}$ by matching up Casimir operators as in 5 of Theorem 3.7. This time, though, we work with respect to the split torus of $SL(2)$ (which is infinitesimally generated by h), as opposed to the compact torus (infinitesimally generated by K). When convenient, we use the "coordinates" (z_1, z_2, z_3) from 4.2. This allows us to view a function $F(M)$ on $X_{\mathbf{R}}$ as a function $F(z_1, z_2, z_3)$ on $\mathbf{C}^3/\mathbf{T}$; i.e. for $\zeta \in \mathbf{T}$, $F(\zeta z_1, \zeta z_2, \zeta z_3) = F(z_1, z_2, z_3)$. Let us give the action of $\mathfrak{sl}(2)$. The formulas are similar to those of the Fock-Bargmann model in the previous section.

$$r(e)F(M) = \pi i \operatorname{tr} M^2 F(M); \tag{4.6}$$

$$r(h)F = (2 + \sum_{j=1}^{3} z_j \frac{\partial}{\partial z_j})F = (2 + \sum_{j=1}^{3} \overline{z}_j \frac{\partial}{\partial \overline{z}_j})F; \tag{4.7}$$

$$r(f)F = \frac{i}{4\pi \operatorname{tr} M} \sum_{j=1}^{3} \frac{\partial^2 F}{\partial z_j \partial \overline{z}_j} = \frac{i}{4\pi(|z_1|^2 + |z_2|^2 + |z_3|^2)} \sum_{j=1}^{3} \frac{\partial^2 F}{\partial z_j \partial \overline{z}_j}. \tag{4.8}$$

THEOREM 4.2. *The above formal representation on $L^2(X_{\mathbf{R}})$ arises from a genuine unitary representation of $SL(2,\mathbf{R}) \times G$, which decomposes into*

$$L^2(X_{\mathbf{R}}) \cong \widehat{\bigoplus_{k \geq 0}} DS_{k+2} \otimes V_{k\rho}. \tag{4.9}$$

Here $V_{k\rho}$ is, as before, the $(k+1)^3$-dimensional representation of $G = U(3)$, with highest weight $k\rho$. By a slight abuse of notation we have written DS_{k+2} to mean the unitary discrete series representation of $SL(2,\mathbf{R})$, instead of its Harish-Chandra module. The decomposition 4.9 implies that the images $r(Z(\mathfrak{sl}(2)))$ and $r(Z(\mathfrak{g}))$ coincide on the Harish-Chandra module of $L^2(X_{\mathbf{R}})$, as in Theorem 3.7.

PROOF. Ignoring the zero matrix which does not affect $L^2(X_{\mathbf{R}})$, we have the "polar decomposition"

$$X_{\mathbf{R}} - \{0\} \cong S \times (0, +\infty), \tag{4.10}$$

where

$$S = \{M \in X_{\mathbf{R}} \mid \operatorname{tr} M = 1\}, \tag{4.11}$$

and we identify $(t, M) \in (0, +\infty) \times S$ with the matrix $tM \in X_{\mathbf{R}}$. Note that S is isomorphic to the complex projective plane, because in (z_1, z_2, z_3) coordinates,

$$S = \{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid \sum_{j=1}^{3} |z_j|^2 = 1\}/\mathbf{T}. \tag{4.12}$$

Moreover, $S \cong G/G_0$, where $G_0 \cong U(2) \times U(1)$ is the subgroup of $U(3)$ stabilizing the line through $(0,0,1)$. (In the classical case, if G were $O(2n)$, this would be analogous to the polar decomposition $\mathbf{R}^{2n} - \{0\} \cong S^{2n-1} \times (0, +\infty)$, where $S^{2n-1} \cong O(2n)/O(2n-1)$ is the unit sphere in $\mathbf{R}^{2n}$.)

The action of G on $X_{\mathbf{R}}$ is entirely on the factor S, which carries a unique G-invariant measure up to scaling. Moreover, $L^2(X_{\mathbf{R}}) \cong L^2((0, +\infty), t^3\, dt) \otimes L^2(S)$. Now $L^2(S)$ decomposes orthogonally under the action of G into

$$L^2(S) \cong \widehat{\bigoplus_{k \geq 0}} V_{k\rho}. \tag{4.13}$$

(This is analogous to the classical decomposition $L^2(S^{2n-1}) \cong \widehat{\bigoplus}_{k \geq 0} \mathcal{H}_k$ under the action of $O(2n)$.) 4.13 holds because $L^2(G/G_0)$ decomposes by the Peter-Weyl theorem into the sum of those representations of G which have a G_0-fixed covector, each V_λ appearing with multiplicity the dimension of $(V_\lambda^*)^{G_0}$. But G_0 is a spherical subgroup of G (equivalently, (G, G_0) is a Gelfand pair), and the V_λ's involved are precisely the $V_{k\rho}$; see for instance [**K**], Tabelle I, line 2 with $n = 2$ and $m = 1$. In particular, $V_{k\rho} \cong V_{k\rho}^*$ has up to scaling precisely one G_0-fixed vector, as can also be seen from the character of $V_{k\rho}$. Thus we obtain that the $V_{k\rho}$-isotypic subspace of $L^2(X_{\mathbf{R}})$ is $L^2((0,+\infty), t^3\,dt)\otimes V_{k\rho}$. Call this subspace E_k; it is stable under $SL(2)\times G$. We claim that $E_k \cong DS_{k+2}\otimes V_{k\rho}$. This will imply both the decomposition 4.9 and the fact that our representation is unitary.

Let us first investigate the formal action r of $\mathfrak{sl}(2)$ on this subspace. By virtue of our having matched up the action of the Casimir operators $C_{\mathfrak{sl}(2)}$ and $C_{\mathfrak{g}}$, we know that $C_{\mathfrak{sl}(2)}$ acts on E_k by the scalar $(k{+}2)k$. Now in "polar coordinates" (t,σ) on $X_{\mathbf{R}}$ (where $t\in(0,+\infty)$ and $\sigma\in S$), $r(e)$ acts by multiplication by $\pi i t^2$, and $r(h)$ by the operator $2+t\frac{\partial}{\partial t}$. Before computing $r(f)$ explicitly on E_k, let us change coordinates to make this look more like the model for DS_{k+2} described in 2.10 through 2.13 and Proposition 2.5. We can explicitly view $V_{k\rho}$ as the space spanned by the G-translates of the function $(z_1\overline{z}_3)^k$, viewed as a function on S. Let $p_S(\sigma)$ be a linear combination of such translates, where $\sigma\in S$. p_S extends to a polynomial function $p(M)$ on $X_{\mathbf{R}}$ (namely to a linear combination of translates of the function $(z_1\overline{z}_3)^k$, viewed as a function on $X_{\mathbf{R}}$; this is a polynomial in the entries of M). Specifically, $p(t\sigma)=t^k p_S(\sigma)$, where $t\in(0,+\infty)$ and $\sigma\in S$. Hence we can view E_k as the space of functions of the form $F(M)=p(M)\varphi(\operatorname{tr}M^2)$, or in "polar coordinates" $(t,\sigma)\in(0,+\infty)\times S$, $F(t\sigma)=p_S(\sigma)\cdot t^k\varphi(t^2)$. So our change of coordinates amounts to taking functions φ such that $t^k\varphi(t^2)\in L^2((0,+\infty),t^3\,dt)$. In other words, writing $u=t^2\in(0,+\infty)$, we are using functions $\varphi(u)\in L^2((0,+\infty),u^{k+1}\,du)$. The upshot of all this is that we obtain

$$E_k = L^2((0,+\infty),u^{k+1}\,du)\otimes V_{k\rho} \tag{4.14}$$

We are now considering $V_{k\rho}$ explicitly as a space of polynomial functions $p(M)$ on all of $X_{\mathbf{R}}$ (not just on S), of degree k in the entries of M. We immediately obtain the effect of $\mathfrak{sl}(2)$ on a function of the form $p(M)\varphi(u)$, where $p\in V_{k\rho}\subset L^2(X_{\mathbf{R}})$, and $u=\operatorname{tr}M^2$. We obtain the following formulas (the action of $\mathfrak{sl}(2)$ only affects the function φ):

$$r(e)p(M)\varphi(u) = p(M)\cdot \pi i u\varphi(u); \tag{4.15}$$

$$r(h)p(M)\varphi(u) = p(M)\cdot (k+2+2u\frac{\partial}{\partial u})\varphi(u); \tag{4.16}$$

$$r(f)p(M)\varphi(u) = p(M)\cdot \frac{i}{\pi}(u\frac{\partial^2}{\partial u^2}+(k+2)\frac{\partial}{\partial u})\varphi(u). \tag{4.17}$$

Here we can deduce 4.17 from 4.8 directly, or from knowing that $C_{\mathfrak{sl}(2)}$ acts by $(k+2)k$, and using Lemma 2.2.

Comparing 4.15–4.17 to 2.10–2.13, we immediately obtain that $E_k\cong DS_{k+2}\otimes V_{k\rho}$. Here the action of $\mathfrak{sl}(2)$ is entirely on the first factor, and is expressed in terms of the above formulas (which depend on k, and are exactly the same as those for r_k from Section 2). We note that the Harish-Chandra module of our realization of DS_{k+2} was on functions of the form $\varphi(u)=q(u)e^{-\pi u}$, where q is a polynomial. Thus functions of the form $p(M)\varphi(u)$ are polynomials in the entries of M, multiplied by the "Gaussian" $e^{-\pi\operatorname{tr}M^2}$. We have essentially identified r by observing its action on such functions. □

We have thus identified the representation r of $\mathfrak{sl}(2)\times G$. Now we know already that the action of $\mathfrak{sl}(2)$ on DS_{k+2} "exponentiates" into a genuine representation of $SL(2)$; so we can use this action to extend our representation R from $B^+\times G$ to all of $SL(2)\times G$. More explicitly, we can define the action of all of $SL(2)$ on functions in E_k, i.e. functions of the form $F(M) = p(M)\varphi(\operatorname{tr} M^2)$, (again, $p(M) \in V_{k\rho}$ is a polynomial):

$$R(\gamma)F(M) = p(M)R_k(\gamma)\varphi(\operatorname{tr} M^2), \qquad \text{for } \gamma \in SL(2). \tag{4.18}$$

Here $R_k(\gamma)$ is given by equations 2.18 and 2.19, and is the explicit "exponentiation" of r_k. Now $L^2(X_{\mathbf{R}}) \cong \widehat{\bigoplus}_{k\geq 0} E_k$, by the analogous decomposition for $L^2(S)$. Hence we have obtained our desired representation R of all of $SL(2) \times G$ on $L^2(X_{\mathbf{R}})$.

We now turn to the question of giving explicit formulas for this representation. Equations 4.4 and 4.5 describe the action of $B^+ \times G$. Now B^+ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ generate $SL(2)$, so it is enough for us to compute the operator $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$. We shall spend the rest of this paper doing so.

We begin by defining $K_k : S \times S \to \mathbf{C}$ to be the integral kernel for the projection operator from $L^2(S)$ to its subspace isomorphic to $V_{k\rho}$. In other words, if $p_S \in L^2(S) \cong \widehat{\bigoplus}_{k\geq 0} V_{k\rho}$, and q_S is the projection of p_S onto the subspace isomorphic to $V_{k\rho}$, then

$$q_S(\sigma) = \int_{\sigma' \in S} p_S(\sigma')K_k(\sigma, \sigma')\, d\sigma'. \tag{4.19}$$

We combine this with 2.19, which describes $R_k(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ on our explicit model for DS_{k+2}, and formally obtain that $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ is given by the following integral operator, expressed in polar coordinates $(t, \sigma) \in (0, +\infty) \times S$:

$$\begin{aligned} R(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})&F(t,\sigma) \\ &= \sum_{k\geq 0} \int_{t'>0} \int_{\sigma' \in S} F(t', \sigma')(-i)^{k+2} \cdot 2\pi \frac{J_{k+1}(2\pi t t')}{tt'} K_k(\sigma, \sigma')\, d\sigma'\, t'^3\, dt. \end{aligned} \tag{4.20}$$

Ignoring matters of convergence, one checks this formally on functions of the form $p_S(\sigma)t^k\varphi(t^2) = p(M)\varphi(u)$, where $p_S \in V_{k\rho} \subset L^2(S)$, and $\varphi \in L^2((0, +\infty), u^{k+1}\, du)$. Hence $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ should be given by integrating against a kernel whose expression in polar coordinates is

$$\frac{-2\pi}{tt'} \sum_{k\geq 0} (-i)^k K_k(\sigma, \sigma') J_{k+1}(2\pi t t'). \tag{4.21}$$

This will be justified once we show in Lemma 4.5 that the above sum converges sufficiently well. We must therefore determine K_k explicitly, and sum the above series.

Proposition 4.3. *Let us view elements $\sigma \in S$ as (Hermitian rank 1) matrices of trace 1. Then*

$$K_k(\sigma, \sigma') = \frac{(k+1)^3}{\operatorname{vol} S} P_k(\operatorname{tr} \sigma\sigma'), \tag{4.22}$$

where

$$P_0(x) = 1, \qquad P_1(x) = \frac{1}{2}(3x - 1), \qquad P_2(x) = \frac{1}{3}(10x^2 - 8x + 1), \ \cdots \tag{4.23}$$

are the orthogonal polynomials on $[0, 1]$ with respect to the measure $(1 - x)\, dx$, normalized so that $P_k(1) = 1$.

Remark 4.4. The P_k are essentially Jacobi polynomials; see (12.12–3) of [**H**]. Their general term is

$$(4.24)\quad P_k(x) = \frac{1}{k+1}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{k+1+j}{j}x^j = \frac{(-1)^k}{k+1}\,{}_2F_1(-k,k+2;1;x).$$

The fact that $P_k(1)=1$ is the Vandermonde theorem ([**H**], (1.4–11)).

Note also that we can view elements of S as belonging to $\mathbf{C}^3/\mathbf{T}$, and represent them by vectors $\vec{z}=(z_1,z_2,z_3)\in\mathbf{C}^3$ with $\sum_{j=1}^3 |z_j|^2 = 1$. Here $\vec{z}$ is determined up to scaling by $\zeta\in\mathbf{T}$; the corresponding matrix is $\sigma = \begin{pmatrix}\overline{z_1}\\ \overline{z_2}\\ \overline{z_3}\end{pmatrix}\begin{pmatrix} z_1 & z_2 & z_3\end{pmatrix}$. Similarly, let σ' be the matrix corresponding to $\vec{w}=(w_1,w_2,w_3)$. Then in these coordinates, the "inner product" $\operatorname{tr}\sigma\sigma'$ is just $|\langle\vec{z},\vec{w}\rangle|^2$, where $\langle\vec{z},\vec{w}\rangle = \sum_{j=1}^3 z_j\overline{w_j}$. We shall feel free to use these coordinates when convenient.

Proof of Proposition 4.3. This is entirely analogous to the theory of zonal spherical harmonics. $(1-x)\,dx$ is the measure on $[0,1]\cong G_0\backslash S\cong G_0\backslash G/G_0$ induced from the measure on S; the identification map $G_0\backslash S\to[0,1]$ is $(z_1,z_2,z_3)\mapsto |z_3|^2$. To be thorough, we recall the proof (see [**F2**], Chapter 2, Section G, especially Theorem (2.55) to Proposition (2.57)).

Let $f_1,\dots f_N$ be an orthonormal basis for $V_{k\rho}\subset L^2(S)$, where $N=\dim V_{k\rho} = (k+1)^3$. Then $K_k(\sigma,\sigma') = \sum_{j=1}^N f_j(\sigma)\overline{f_j(\sigma')}$. Now $K_k(g^{-1}\sigma g, g^{-1}\sigma' g) = K_k(\sigma,\sigma')$ for $g\in G$, because the projection respects the action of G. Viewing $\vec{z}$ and $\vec{w}$ as row vectors, this means that $K_k(\vec{z}g,\vec{w}g) = K_k(\vec{z},\vec{w})$ for $g\in G$; hence $K_k(\vec{z},\vec{w})$ can depend only on $|\langle\vec{z},\vec{w}\rangle|^2$. (We could have made K_k depend only on $\langle\vec{z},\vec{w}\rangle$, but this last quantity is only defined up to a factor of $\zeta\in\mathbf{T}$, so we only care about $|\langle\vec{z},\vec{w}\rangle|$. Using the square of the absolute value is even better, since this is now a polynomial in the entries of the matrices σ and σ'.)

Let us arrange our orthonormal basis so that f_1 spans the one-dimensional space of G_0-fixed vectors in $V_{k\rho}$. (f_1 is analogous to a zonal spherical harmonic.) Also observe that G_0 is the stabilizer of the "north pole" $\vec{n}=(0,0,1)\in\mathbf{C}^3/\mathbf{T}$. Consider the nonzero G_0-invariant linear functional on $V_{k\rho}$, sending p_S to $p_S(\vec{n})$. $V_{k\rho}$ is self-dual, so this functional must be obtained by taking an inner product with a G_0-fixed vector; so for some nonzero constant C, we obtain that

$$(4.25)\qquad p_S(\vec{n}) = C\int_{\sigma\in S} p_S(\sigma)\overline{f_1(\sigma)}\,d\sigma, \qquad \text{for all } p_S\in V_{k\rho}.$$

Testing on f_1, we find that $C=f_1(\vec{n})$. Similarly, $f_2(\vec{n})=\cdots=f_N(\vec{n})=0$. Hence $K_k(\sigma,\vec{n}) = f_1(\sigma)\overline{f_1(\vec{n})}$. Also, for all σ, $K_k(\sigma,\sigma) = K_k(\vec{n},\vec{n}) = |f_1(\vec{n})|^2$, by the G-invariance of K_k. This yields

$$(4.26)\qquad (k+1)^3 = \int_{\sigma\in S} K_k(\sigma,\sigma)\,d\sigma = (\operatorname{vol} S)\,|f_1(\vec{n})|^2.$$

Putting this together we obtain that K_k has the form

$$(4.27)\qquad K_k(\vec{z},\vec{w}) = \frac{(k+1)^3}{\operatorname{vol} S} P_k(|\langle\vec{z},\vec{w}\rangle|^2),$$

for some function P_k; furthermore, by evaluating $K_k(\vec{n},\vec{n})$, we see that $P_k(1)=1$.

We now show that P_k is a polynomial of degree at most k. For $x \in [0,1]$ and arbitrary $\zeta \in \mathbf{T}$, $P_k(x)$ equals a constant multiple of $K_k\big((\sqrt{1-x}, 0, \zeta\sqrt{x}), \vec{n}\big)$, which in turn is a constant multiple of $f_1\big((\sqrt{1-x}, 0, \zeta\sqrt{x})\big)$. But $f_1(\sigma) \in V_{k\rho}$ is a degree k polynomial in the entries of the matrix σ, i.e., $P_k(x)$ is a polynomial of degree k in the expressions x, $1-x$, $\zeta\sqrt{x(1-x)}$, and $\zeta^{-1}\sqrt{x(1-x)}$. But $P_k(x)$ is independent of ζ, and so the latter two expressions can only appear if multiplied by each other. Looking at the degrees of the resulting expressions, we see that $P_k(x)$ is a polynomial in x of degree at most k.

Finally, the P_k's are orthogonal with respect to the measure $(1-x)\,dx$ on $[0,1]$. To see this, note first that for a given k, the G_0-invariant polynomial $f_1(\vec{z}) \in V_{k\rho}$ is a constant multiple of $P_k(|z_3|^2)$. But now, if $k \neq l$, then the two polynomials $P_k(|z_3|^2) \in V_{k\rho}$ and $P_l(|z_3|^2) \in V_{l\rho}$ are orthogonal functions in $L^2(S)$, because 4.13 is an orthogonal decomposition. In coordinates, this is equivalent to

$$\int_{\substack{(z_1,z_2,z_3)\in\mathbf{C}^3 \\ |z_1|^2+|z_2|^2+|z_3|^2=1}} P_k(|z_3|^2)P_l(|z_3|^2)\,d\mu = 0, \qquad \text{if } k \neq l. \tag{4.28}$$

Here $d\mu$ corresponds to the G-invariant measure on S. Up to a constant, it must be the standard measure on the unit sphere in $\mathbf{C}^3$. We evaluate the above integral along subsets where $|z_3|$ is constant to obtain the desired orthogonality property

$$\int_{x=0}^{1} P_k(x)P_l(x)(1-x)\,dx = 0, \qquad \text{if } k \neq l. \tag{4.29}$$

Specifically, change coordinates to

$$\begin{aligned} z_1 &= x_1 + iy_1 = e^{i\alpha}\sin A \sin B, \\ z_2 &= x_2 + iy_2 = e^{i\beta}\sin A \cos B, \\ z_3 &= x_3 + iy_3 = e^{i\gamma}\cos A, \end{aligned} \tag{4.30}$$

and use $d\mu = (1/x_1)dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3$. □

We now work on the sum 4.21. This amounts to evaluating the function $\Sigma(x,y)$, where

$$\Sigma(x,y) = \sum_{k\geq 0} (-i)^k (k+1)^3 P_k(x) J_{k+1}(y), \tag{4.31}$$

for $x \in [0,1]$ and $y \in (0,+\infty)$. We begin with the generating function for the P_k's, obtained from the well-known generating function for the Jacobi polynomials (see for instance [**H**], section 2.5, exercise 7). It is valid for $x \in [0,1]$ and $|q| < 1$, or more generally for x in a compact set C and $|q|$ smaller than a constant depending on C.

$$\sum_{k\geq 0}(k+1)P_k(x)q^k = \frac{2}{A(1-q+A)} = q^{-1}\left(\frac{q-1}{2(1-x)A} + \frac{1}{2(1-x)}\right), \tag{4.32}$$

$$A = \sqrt{1 + 2q(1-2x) + q^2}. \tag{4.33}$$

The branch of the square root is the one that equals 1 when $q = 0$.

Lemma 4.5. *Σ is an entire function of x and y.*

Proof. By (4.5–9) of [**H**], $|J_n(y)| \le |y|^n e^n (2n)^{-n} e^{|y|}$. So for y in a compact set, and for every $M_0 > 0$, we can estimate $|J_{k+1}(y)| \le M_0^{-k}$ once k is sufficiently large (depending on the compact set and on M_0). We can also bound $P_k(x)$ in terms of k: given a compact $C \in \mathbf{C}$, the range of validity of 4.32 implies the existence of $M_1 > 0$ such that $x \in C \Rightarrow |P_k(x)| \le M_1^k$. Thus the sum in Σ converges absolutely and uniformly on compact sets. (A specific case, which will be useful later, is that $P_k(x)$ is $O((1+\epsilon)^k)$ uniformly on $[0,1]$ — this is because for $x \in [0,1]$, the poles of the generating function are on the unit circle $\{q \mid |q| = 1\}$.) □

Remark 4.6. As mentioned before, this lemma justifies the manipulations that we perform in constructing the integral operator for $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$. Namely, the integral expression that we shall write down for $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))F$ will be correct if F is a smooth function on $X_{\mathbf{R}}$ with compact support. More precisely, the estimates in Lemma 4.5 imply that the integral of the sum in 4.21 against a smooth F of compact support can be evaluated term by term. (Only the values of $\Sigma(x,y)$ for bounded y come into play.) Let F_k be the projection of F onto E_k. One sees that the integral of F against the kth term in 4.21 is indeed $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))F_k$, because integrating out σ' first gives the projection of F to $F_k \in E_k$. Then the remaining integral in t' is correct because we explicitly know the action of $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ on F_k. Thus our term-by-term integration yields that $\sum_k R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))F_k$ converges pointwise to the integral of F against the proposed kernel. On the other hand, in terms of the L^2 norm, $\|F\|^2 = \sum_k \|F_k\|^2 < \infty$, and moreover $\left\|R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))F_k\right\| = \|F_k\|$; so $\sum_k R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))F_k$ converges in L^2 to $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))F$. But if a sequence of functions converges both in L^2 and pointwise, then the L^2 limit equals the pointwise limit, because L^2 convergence of a sequence implies that some subsequence converges pointwise to the L^2 limit. Hence the result of integrating F against the proposed kernel is indeed $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))F$, if F has compact support. Since $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ is unitary, its extension to all of L^2 is now straightforward.

We are now in a position to evaluate $\Sigma(x,y)$.

Proposition 4.7.

$$\Sigma(x,y) = \frac{y}{2} e^{iy} {}_1F_1(3/2; 1; -2ixy), \tag{4.34}$$

where ${}_1F_1$ is the confluent hypergeometric function

$$ {}_1F_1(a;b;t) = 1 + \frac{a}{b}\frac{t}{1} + \frac{a(a+1)}{b(b+1)}\frac{t^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{t^3}{3!} + \cdots . \tag{4.35}$$

Proof. Without loss of generality, we can assume that $0 < x \le 1$ and that y lies in a compact set. The general result will then follow by analytic continuation. Our estimates on $\Sigma(x,y)$ show that

$$\Sigma(x,y) = \lim_{r \to 1^-} r \sum_{k \ge 0} (k+1)^3 (-ir)^k P_k(x) J_{k+1}(y/r). \tag{4.36}$$

(For $1/2 < r < 1$, some tail $\sum_{k>N}$ of the above sum is small uniformly in r. Then use the continuity of the first N terms.) Using the integral expression for J_{k+1} from 2.20, we obtain

$$\Sigma(x,y) = \lim_{r \to 1^-} \frac{i}{2\pi} \int_{\theta=0}^{2\pi} \sum_{k \ge 0} (k+1)^3 (re^{i\theta})^{k+1} P_k(x) e^{-i(y/r)\cos\theta}\, d\theta. \tag{4.37}$$

Interchanging the summation with the integration is justified by our estimates on the size of P_k and J_{k+1} in terms of k. For convenience, write $q = re^{i\theta}$. To evaluate the sum $\sum_{k\geq 0}(k+1)^3 P_k(x)q^{k+1}$, we apply the differential operator $q\frac{\partial}{\partial q}q\frac{\partial}{\partial q}q$ to the identity 4.32. We obtain

$$\sum_{k\geq 0}(k+1)^3 P_k(x)q^{k+1} = \frac{q(1-q)((1+q)^2+2qx)}{((1+q)^2-4qx)^{5/2}}, \tag{4.38}$$

where the denominator is just A^5. Since $q = re^{i\theta}$, $(\cos\theta)/r = (1/2)(1/q + q/r^2)$. Hence $\Sigma(x,y)$ equals the limit, as $r \to 1^-$, of

$$\frac{i}{2\pi}\int_{\theta=0}^{2\pi} \frac{q(1-q)((1+q)^2+2qx)}{((1+q)^2-4qx)^{5/2}} e^{-i(y/2)(1/q+q+q(1/r^2-1))}\, d\theta. \tag{4.39}$$

Now make the change of variable $b = (1+q)^2/q = 1/q + q + 2$, $db = i((q^2-1)/q)\, d\theta$. As θ traverses $[0, 2\pi]$, b moves *clockwise* around a thin and flat ellipse Ω_r in the complex plane, enclosing the real line segment $[0, 4]$. With some care about the branches of the square roots (or by checking the limit as x approaches 0) we obtain $\Sigma(x,y)$ as the limit as $r \to 1^-$ of

$$\frac{-1}{2\pi}\int_{b \text{ on } \Omega_r} \frac{b+2x}{(b-4x)^2\sqrt{b(b-4x)}} e^{-i(y/2)(b-2+q(1/r^2-1))}\, db, \tag{4.40}$$

where $q = (b - 2 - \sqrt{b(b-4)})/2$. We have chosen the branches of $\sqrt{b(b-4x)}$ and of $\sqrt{b(b-4)}$ for $b \in \mathbf{C} - [0,4]$, by requiring that if b is real and greater than 4, then the above square roots will be positive. At this point we note that the integral is independent of the path Ω_r, and so we can replace Ω_r by, say, $\Omega_{1/2}$, and *then* let r tend to 1. Hence

$$\Sigma(x,y) = \frac{-e^{iy}}{2\pi}\int_{b \text{ on } \Omega_{1/2}} \frac{b+2x}{(b-4x)^2\sqrt{b(b-4x)}} e^{-iyb/2}\, db. \tag{4.41}$$

This last integral now depends on x in a simple way, and so we can replace b with xb. The new contour goes clockwise around the segment $[0, 4/x]$, but can again be replaced with $\Omega_{1/2}$. We obtain

$$\Sigma(x,y) = \frac{-e^{iy}}{2\pi x}\int_{b \text{ on } \Omega_{1/2}} \frac{b+2}{(b-4)^2\sqrt{b(b-4)}} e^{-ixyb/2}\, db \tag{4.42}$$

$$= \frac{-e^{iy}}{2\pi x}\sum_{n\geq -1}\frac{(-ixy/2)^{n+1}}{(n+1)!}\int_{b \text{ on } \Omega_{1/2}} \frac{(b+2)b^{n+1}}{(b-4)^2\sqrt{b(b-4)}}\, db, \tag{4.43}$$

where we recall that b traverses $\Omega_{1/2}$ *clockwise.* (We have begun the summation with $n = -1$ because the first term turns out to vanish.) One easily evaluates this last integral in terms of the "residue at ∞" (i.e., change variables so that $1/b$ is the variable of integration). The value of the integral turns out to be $-2\pi i(n+1)(3/2)_n 4^n/n!$, which vanishes for $n = -1$. (For $n \geq 1$, we have used the Pochhammer symbol $(3/2)_n = (3/2)(3/2+1)\cdots(3/2+n-1)$. For $n = 0$, we take $(3/2)_0 = 1$.) Proposition 4.7 finally follows. □

REMARK 4.8. Once we obtain 4.42, we know that the sum $\Sigma(x,y)$ has the form $(e^{iy}/x)f(xy)$ for some function f. Then another way to find that f is the stated confluent hypergeometric function is to observe that $R(\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right))r(e) = -r(f)R(\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right))$, which leads to an easily solvable differential equation for f. This is analogous to Remark 2.7.

We put together all these ingredients to obtain

THEOREM 4.9. *In polar coordinates,* $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ *acts on* $F \in \mathcal{C}_c^\infty(X_{\mathbf{R}})$ *by*

$$(4.44) \quad R(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})F(t,\sigma) = \frac{-2\pi^2}{\operatorname{vol} S} \iint_{\substack{t' \in (0,+\infty) \\ \sigma' \in S}} F(t',\sigma') e^{2\pi i t t'} {}_1F_1(3/2; 1; -2\pi i t t' \operatorname{tr} \sigma\sigma')\, d\sigma'\, t'^3\, dt'.$$

In terms of the matrices $M = t\sigma$ *and* $N = t\sigma'$,

$$(4.45) \quad R(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})F(M) = \frac{-4}{\pi} \int_{N \in X_{\mathbf{R}}} F(N) e^{2\pi i \operatorname{tr} M \operatorname{tr} N} {}_1F_1(3/2; 1; -2\pi i \operatorname{tr} MN)\, dN,$$

with the measure dN *defined in 4.3.*

By Theorems 4.2 and 4.9, our formulas 4.4, 4.5, and 4.45 define a representation of $SL(2) \times U(3)$ on $L^2(X_{\mathbf{R}})$, which decomposes according to

$$(4.46) \quad L^2(X_{\mathbf{R}}) \cong \widehat{\bigoplus_{k \geq 0}} DS_{k+2} \otimes V_{k\rho}.$$

As mentioned in the introduction, this decomposition realizes the adjoint lift from $SL(2,\mathbf{R})$ to $U(3)$, and the formulas are sufficiently explicit that they might possibly generalize to the p-adic case. This would involve addressing several points. The first point is that there are no anisotropic forms of $U(3)$ over a p-adic field. Moreover, for global applications one would need to deal with the split case anyhow. It seems reasonable to first try out the split case over $\mathbf{R}$, using the techniques of this paper. These techniques easily give rise to a formal representation of $\mathfrak{sl}(2,\mathbf{R}) \times \mathfrak{gl}(3,\mathbf{R})$ on the space of functions on a "split" form X'_{split} of X', namely the 3×3 real matrices M of rank at most 1. (It is nonetheless unclear whether e.g., restricting $\operatorname{tr} M$ to be nonnegative provides a suitable "half" of this space to obtain an analog of X.) In the terminology of Remark 3.9, we then obtain a formal representation of $\mathfrak{sl}(2) \times GL(3)$ on the space of functions on X'_{split}. We merely begin with the representation of $B^+ \times GL(3)$ defined by the same formulas as 4.4 and 4.5. We can then match up the Casimir operators $C_{\mathfrak{sl}(2)}$ and $C_{\mathfrak{gl}(3)}$, and use Lemma 2.2, to obtain a formal action of all of $\mathfrak{sl}(2)$. It seems more difficult to "exponentiate" this resulting representation to all of $SL(2)$; the main complication is that now the structure of $L^2(X'_{\text{split}})$ as a representation of $GL(3)$ seems less transparent now that the group is no longer compact. Even assuming some analog of Theorem 4.2, telling us that the formal representation comes from a genuine representation of $SL(2)$, we expect to run into some hairy calculations in trying to generalize our computation of $\Sigma(x,y)$. Essentially, the sum in 4.31 will get replaced by an integral, because the analog of $L^2(S)$ will now decompose continuously under $GL(3)$. Instead of trying to evaluate such an integral, it may be worth first investigating if the formula 4.45 still works. This would be analogous to the fact that for the dual pair $SL(2) \times O(2n)$, the action of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is essentially via a Fourier transform no matter what the signature of the quadratic form fixed by $O(2n)$.

The second complication that arises in extending our results to p-adic fields is how to translate our formulas for $\mathbf{R}$ to a p-adic setting. It is straightforward to generalize the definition of $X'_{\mathbf{R}}$, whether we are dealing with $GL(3)$ or with $U(3)$ over a p-adic field. Taking "half" of the resulting space seems more mysterious. Perhaps in the case of a unitary group over a p-adic field F, one could require the trace of M to belong to the index 2 subgroup of $F^{\times}$, consisting of norms from the quadratic extension of F used to construct the unitary group. In that case, the analog of the action of $B^{+} \times G$ is easy to write down, using an additive character $e_F : F \to \mathbf{C}^{\times}$ instead of the function $e(x) = e^{2\pi i x}$. The next step, which is slightly more difficult, is to write down (or guess) a formula giving the action of $R(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ in terms of an integral kernel. (X'_{split} still carries more or less the same measure as before.) In this respect the expression $e^{2\pi i \operatorname{tr} M \operatorname{tr} N}$ in the integral should correspond to $e_F(\operatorname{tr} M \operatorname{tr} N)$, but the confluent hypergeometric function seems less tractable. Of course, it would be a difficult calculation to verify that the resulting formulas, even if correct, do indeed fit together to yield a genuine representation of $SL(2) \times G$.

Even if these complications are overcome, and we obtain representations of (say) $SL(2) \times GL(3)$ over $\mathbf{R}$ and every $\mathbf{Q}_p$, that yield the local adjoint lift, it will probably still be a difficult problem to show that these representations are globally coherent, in the sense that they match up *automorphic* representations of $SL(2)$ with *automorphic* representations of $GL(3)$. Again bearing in mind the case of $SL(2) \times O(2n)$, one could hope that there might exist some analog of the Poisson summation formula. Such an analog might well be rather difficult to find. Another direction for further investigation is to search for explicit constructions in the style of Section 4 for other groups G. Even if one sticks to compact G, it may be nontrivial to find a suitable X and $X_{\mathbf{R}}$, since Example 3.11 seems too simplistic a construction, and may not yield suitable formulas. An even more serious problem in generalizing Section 4 to other G is the fact that our "polar decomposition," 4.10, is rather special in that the representations of G occurring in $L^2(S)$ are all ones with highest weight a multiple of a single weight, ρ. This is one of the reasons why our calculations in this paper remained tractable.

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